Python Tools for Analyzing Linear Systems

CDS 110/ChE 105, Winter 2024
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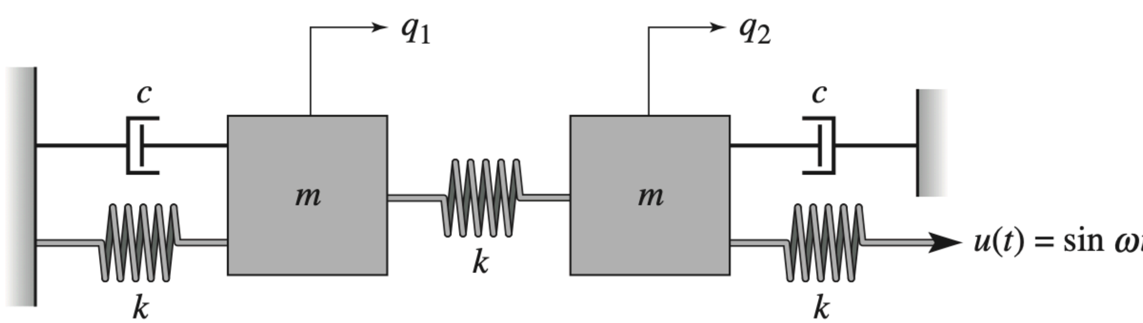
In this lecture we describe tools in the Python Control Systems Toolbox (python-control) that can be used to analyze linear systems, including some of the options available to present the information in different ways.

In [1]:
```
import numpy as np
%matplotlib inline
import matplotlib.pyplot as plt

try:
  import control as ct
  print("python-control version:", ct.__version__)
except ImportError:
  # Version 0.10.0 is enough for this notebook
  !pip install control
  import control as ct
```

```
python-control version: 0.10.0
```

Coupled mass spring system

Consider the spring mass system below:



We wish to analyze the time and frequency response of this system using a variety of python-control functions for linear systems analysis.

System dynamics

The dynamics of the system can be written as

$$
\begin{aligned}
m\ddot{q}_1 &= -2kq_1 - c\dot{q}_1 + kq_2, \\
m\ddot{q}_2 &= kq_1 - 2kq_2 - c\dot{q}_2 + ku
\end{aligned}
$$

or in state space form:

$$\frac{dx}{dt} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{2k}{m} & \frac{k}{m} & -\frac{c}{m} & 0 \\ \frac{k}{m} & -\frac{2k}{m} & 0 & -\frac{c}{m} \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \frac{k}{m} \end{bmatrix} u.$$

In [2]:
```
# Define the parameters for the system
m, c, k = 1, 0.1, 2
# Create a linear system
A = np.array([
    [0, 0, 1, 0],
    [0, 0, 0, 1],
    [-2*k/m, k/m, -c/m, 0],
    [k/m, -2*k/m, 0, -c/m]
])
B = np.array([[0], [0], [0], [k/m]])
C = np.array([[1, 0, 0, 0], [0, 1, 0, 0]])
D = 0

sys = ct.ss(A, B, C, D, outputs=['q1', 'q2'], name="coupled spring mass")
print(sys)
```

```
<StateSpace>: coupled spring mass
Inputs (1): ['u[0]']
Outputs (2): ['q1', 'q2']
States (4): ['x[0]', 'x[1]', 'x[2]', 'x[3]']

A = [[ 0.   0.   1.   0. ]
     [ 0.   0.   0.   1. ]
     [-4.   2.  -0.1  0. ]
     [ 2.  -4.   0.  -0.1]]

B = [[0.]
     [0.]
     [0.]
     [2.]]

C = [[1. 0. 0. 0.]
     [0. 1. 0. 0.]]

D = [[0.]
     [0.]]
```

Another way to get these same dynamics is to define and input/output system:

In [3]:
```
coupled_params = {'m': 1, 'c': 0.1, 'k': 2}
def coupled_update(t, x, u, params):
  m, c, k = params['m'], params['c'], params['k']
  return np.array([
      x[2], x[3],
      -2*k/m * x[0] + k/m * x[1] - c/m * x[2],
```

```python
        k/m * x[0] -2*k/m * x[1] - c/m * x[3] + k/m * u[0]
    ])
def coupled_output(t, x, u, params):
    return x[0:2]
coupled = ct.nlsys(
    coupled_update, coupled_output, inputs=1, outputs=['q1', 'q2'],
    states=['q1', 'q2', 'q1dot', 'q2dot'], name='coupled (nl)',
    params=coupled_params
)
print(coupled.linearize([0, 0, 0, 0], [0]))
```

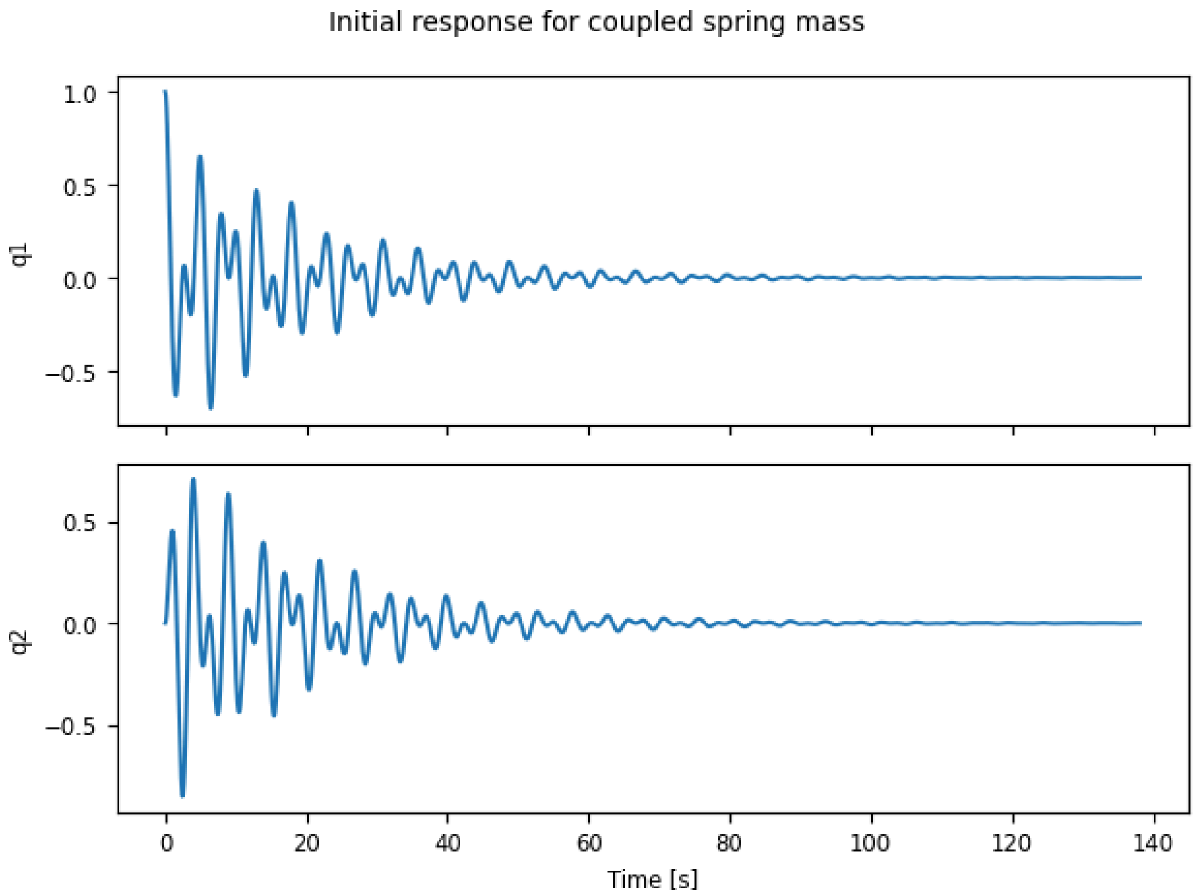
```
<StateSpace>: sys[0]
Inputs (1): ['u[0]']
Outputs (2): ['y[0]', 'y[1]']
States (4): ['x[0]', 'x[1]', 'x[2]', 'x[3]']

A = [[ 0.    0.    1.    0. ]
     [ 0.    0.    0.    1. ]
     [-4.    2.   -0.1   0. ]
     [ 2.   -4.    0.   -0.1]]

B = [[0.]
     [0.]
     [0.]
     [2.]]

C = [[1. 0. 0. 0.]
     [0. 1. 0. 0.]]

D = [[0.]
     [0.]]
```

Initial response

The `initial_response` function can be used to compute the response of the system with no input, but starting from a given initial condition. This function returns a response object, we can be used for plotting.

In [4]:
```python
response = ct.initial_response(sys, X0=[1, 0, 0, 0])
response.plot()
```

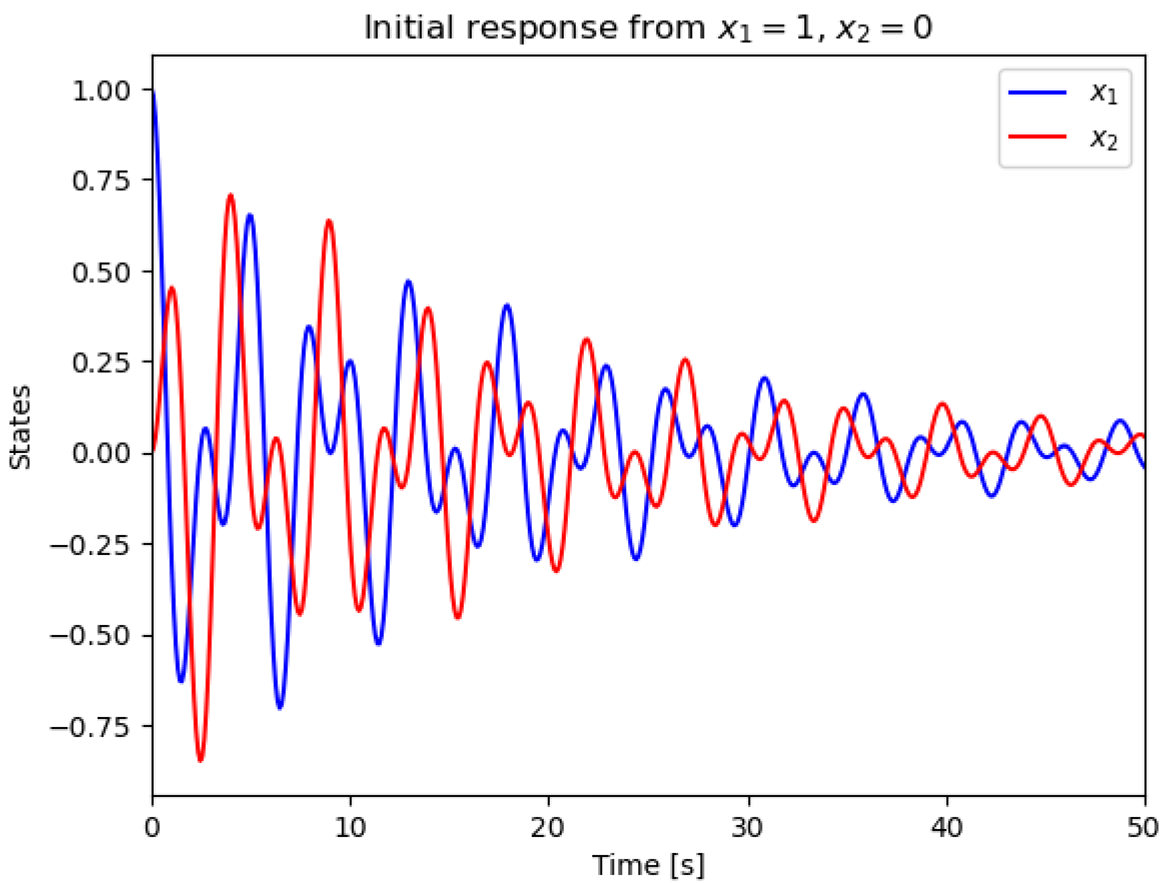
Out[4]:
```
array([[list([<matplotlib.lines.Line2D object at 0x11851a240>])],
       [list([<matplotlib.lines.Line2D object at 0x11b6cf110>])]],
      dtype=object)
```



If you want to play around with the way the data are plotted, you can also use the response object to get direct access to the states and outputs.

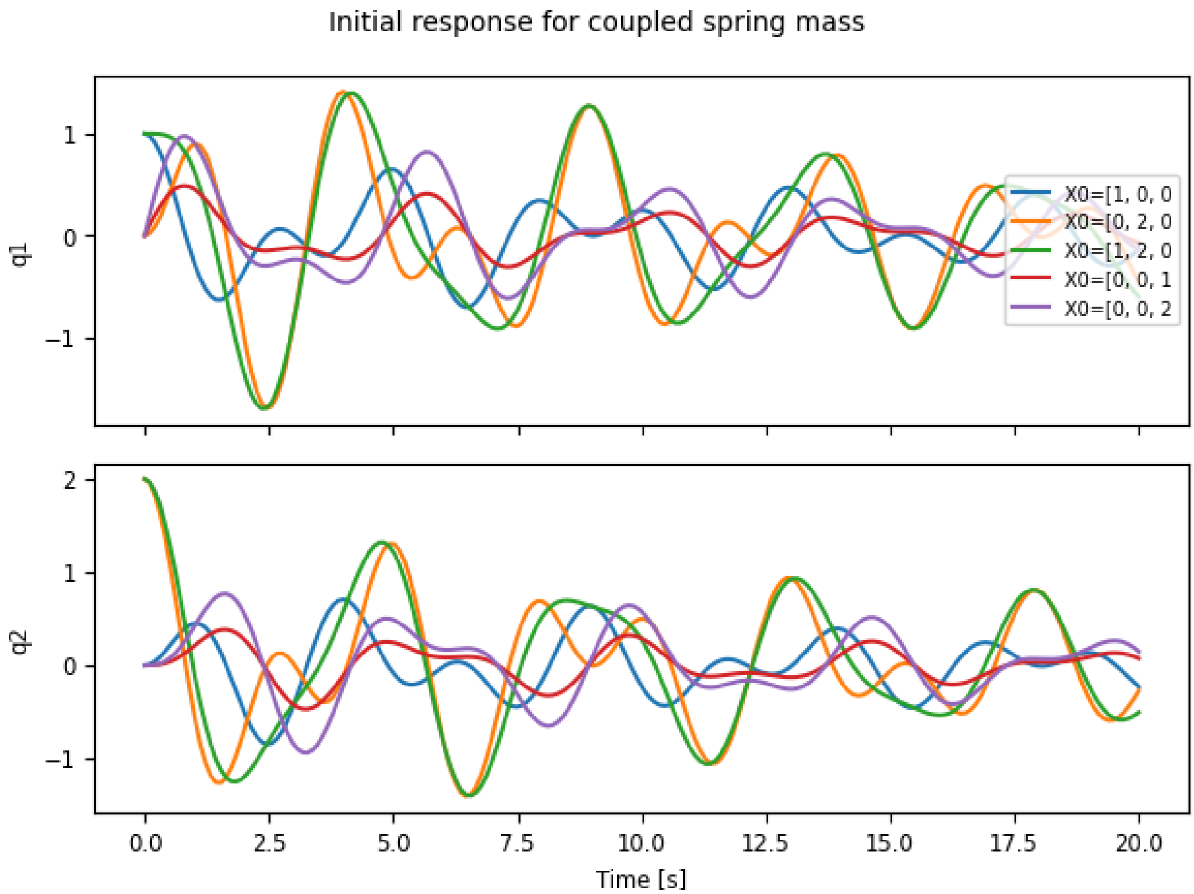
In [5]:
```
# Plot the outputs of the system on the same graph, in different colors
?ct.TimeResponseData
t = response.time
x = response.states
plt.plot(t, x[0], 'b', t, x[1], 'r')
plt.legend(['$x_1$', '$x_2$'])
plt.xlim(0, 50)
plt.ylabel('States')
plt.xlabel('Time [s]')
plt.title("Initial response from $x_1 = 1$, $x_2 = 0$")
```

Out[5]:
```
Text(0.5, 1.0, 'Initial response from $x_1 = 1$, $x_2 = 0$')
```



There are also lots of options available in `initial_response` and `.plot()` for tuning the plots that you get.

In [6]:
```
# Do some Python magic to get different colors
from itertools import cycle
prop_cycle = plt.rcParams['axes.prop_cycle']
colors = cycle(prop_cycle.by_key()['color'])

for X0 in [[1, 0, 0, 0], [0, 2, 0, 0], [1, 2, 0, 0], [0, 0, 1, 0], [0, 0, 2,
  response = ct.initial_response(sys, T=20, X0=X0)
  response.sysname = f"{X0=}" # Hack to label the plots differently
  response.plot(color=next(colors))
```

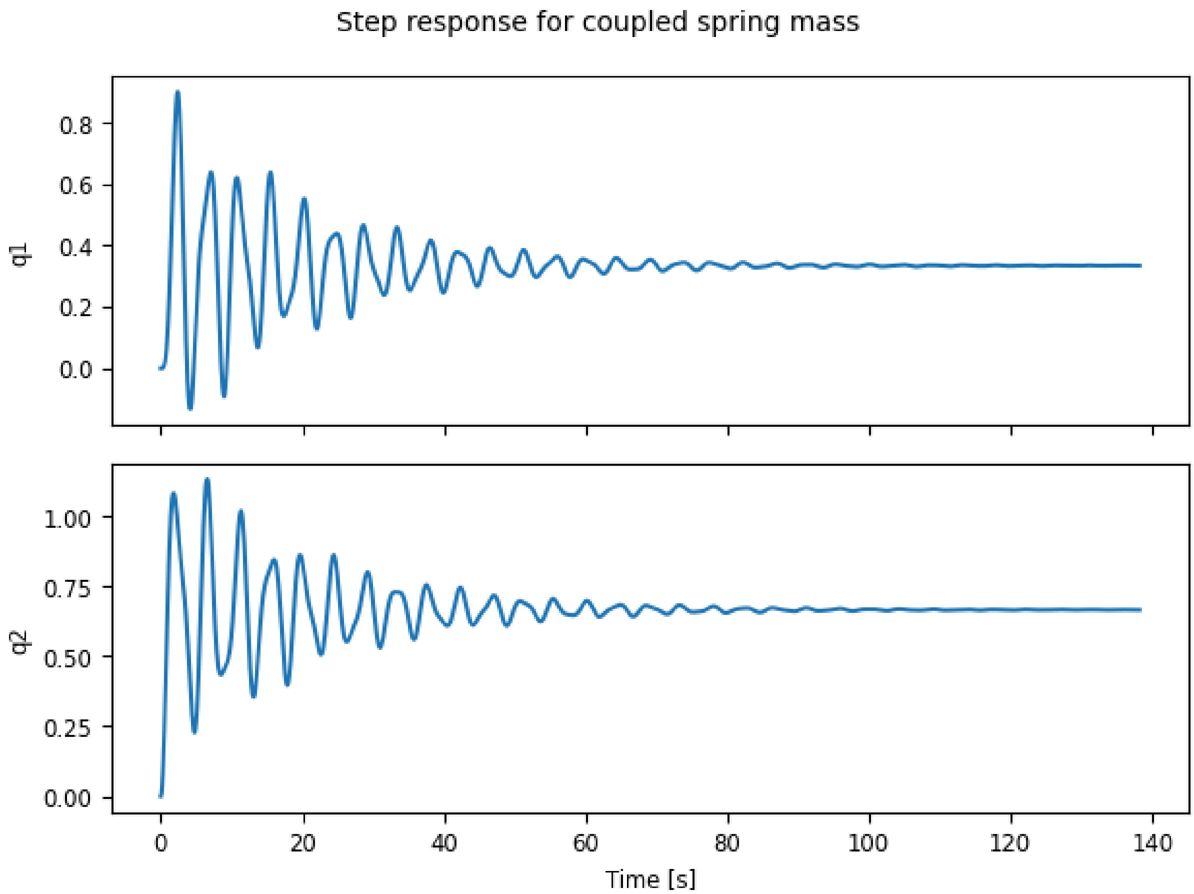


Step response

Similar to `initial_response`, you can also generate a step response for a linear system using the `step_response` function, which returns a time response object:

In [7]:
```
ct.step_response(sys).plot()
```

Out[7]:
```
array([[list([<matplotlib.lines.Line2D object at 0x11bd60a70>])],
       [list([<matplotlib.lines.Line2D object at 0x11bdac7a0>])]],
      dtype=object)
```



We can analyze the properties of the step response using the `stepinfo` command:

In [8]:
```
step_info = ct.step_info(sys)
print("Input 0, output 0 rise time = ",
      step_info[0][0]['RiseTime'], "seconds\n")
step_info
```

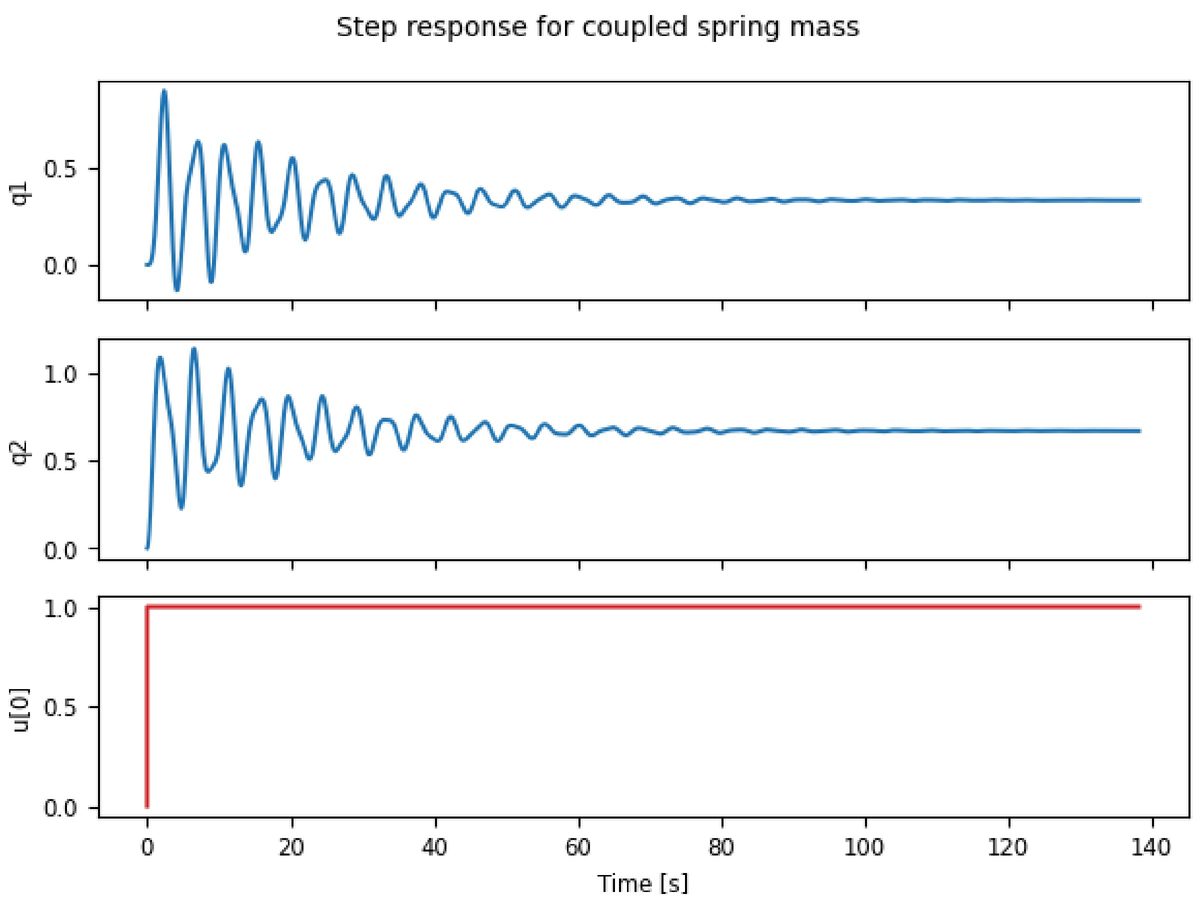
```
Input 0, output 0 rise time =  0.6153902252990775 seconds
```

Out[8]:
```
[[{'RiseTime': 0.6153902252990775,
   'SettlingTime': 89.02645259326653,
   'SettlingMin': -0.13272845655369417,
   'SettlingMax': 0.9005994876222034,
   'Overshoot': 170.17984628666102,
   'Undershoot': 39.81853696610825,
   'Peak': 0.9005994876222034,
   'PeakTime': 2.3589958636464634,
   'SteadyStateValue': 0.33333333333333337}],
 [{'RiseTime': 0.6153902252990775,
   'SettlingTime': 73.6416969607896,
   'SettlingMin': 0.2276019820782241,
   'SettlingMax': 1.13389337710215,
   'Overshoot': 70.08400656532254,
   'Undershoot': 0,
   'Peak': 1.13389337710215,
   'PeakTime': 6.564162403190159,
   'SteadyStateValue': 0.6666666666666665}]]
```

Note that by default the inputs are not included in the step response (since they are a bit boring), but you can change that:

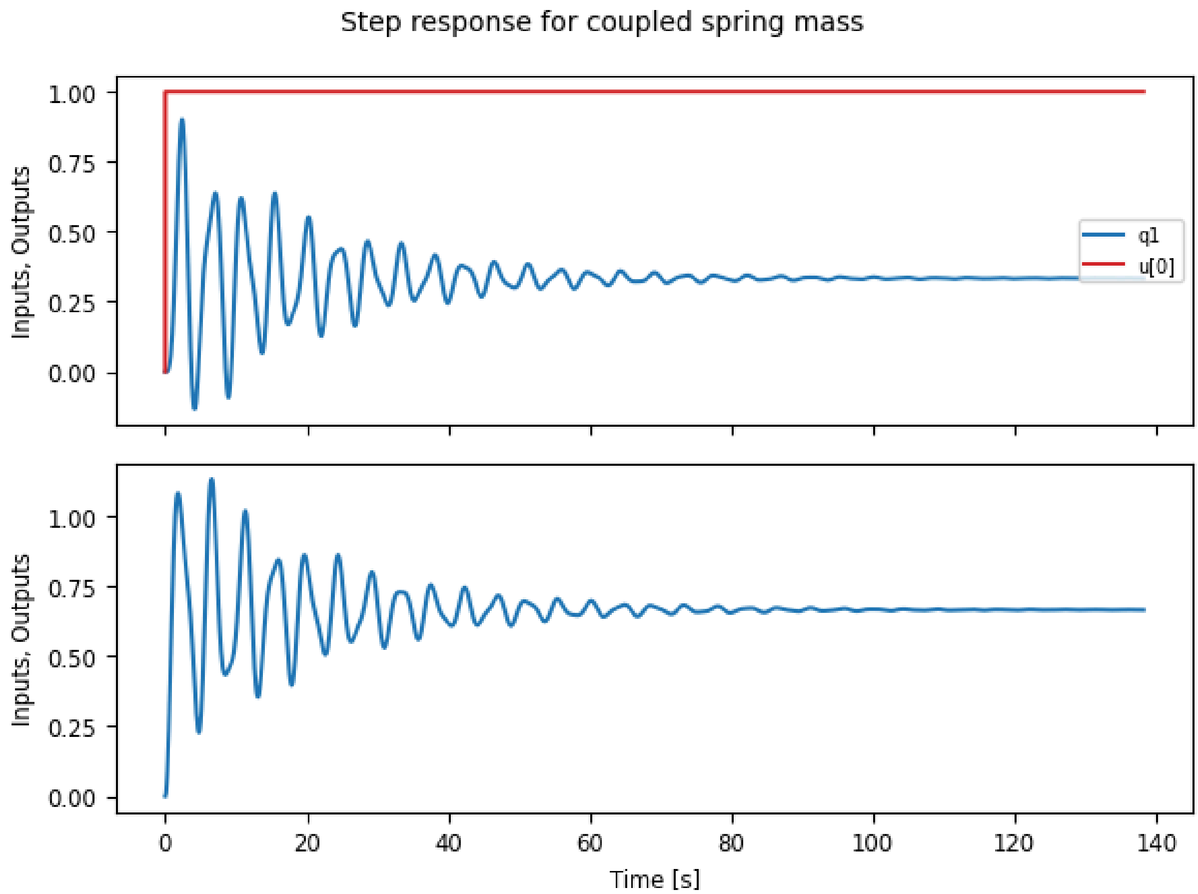
In [9]:
```python
stepresp = ct.step_response(sys)
stepresp.plot(plot_inputs=True)
```

Out[9]:
```
array([[list([<matplotlib.lines.Line2D object at 0x11bead6a0>])],
       [list([<matplotlib.lines.Line2D object at 0x11bead7f0>])],
       [list([<matplotlib.lines.Line2D object at 0x11beadac0>])]],
      dtype=object)
```



In [10]:
```python
stepresp.plot(plot_inputs='overlay')
```

Out[10]:
```
array([[list([<matplotlib.lines.Line2D object at 0x11bef6cf0>, <matplotlib.
lines.Line2D object at 0x11bf952b0>])],
       [list([<matplotlib.lines.Line2D object at 0x11bf950a0>])]],
      dtype=object)
```



In [11]:
```
# Look at the "shape" of the step response
print(f"{stepresp.time.shape=}")
print(f"{stepresp.inputs.shape=}")
print(f"{stepresp.states.shape=}")
print(f"{stepresp.outputs.shape=}")
```

```
stepresp.time.shape=(1348,)
stepresp.inputs.shape=(1, 1, 1348)
stepresp.states.shape=(4, 1, 1348)
stepresp.outputs.shape=(2, 1, 1348)
```

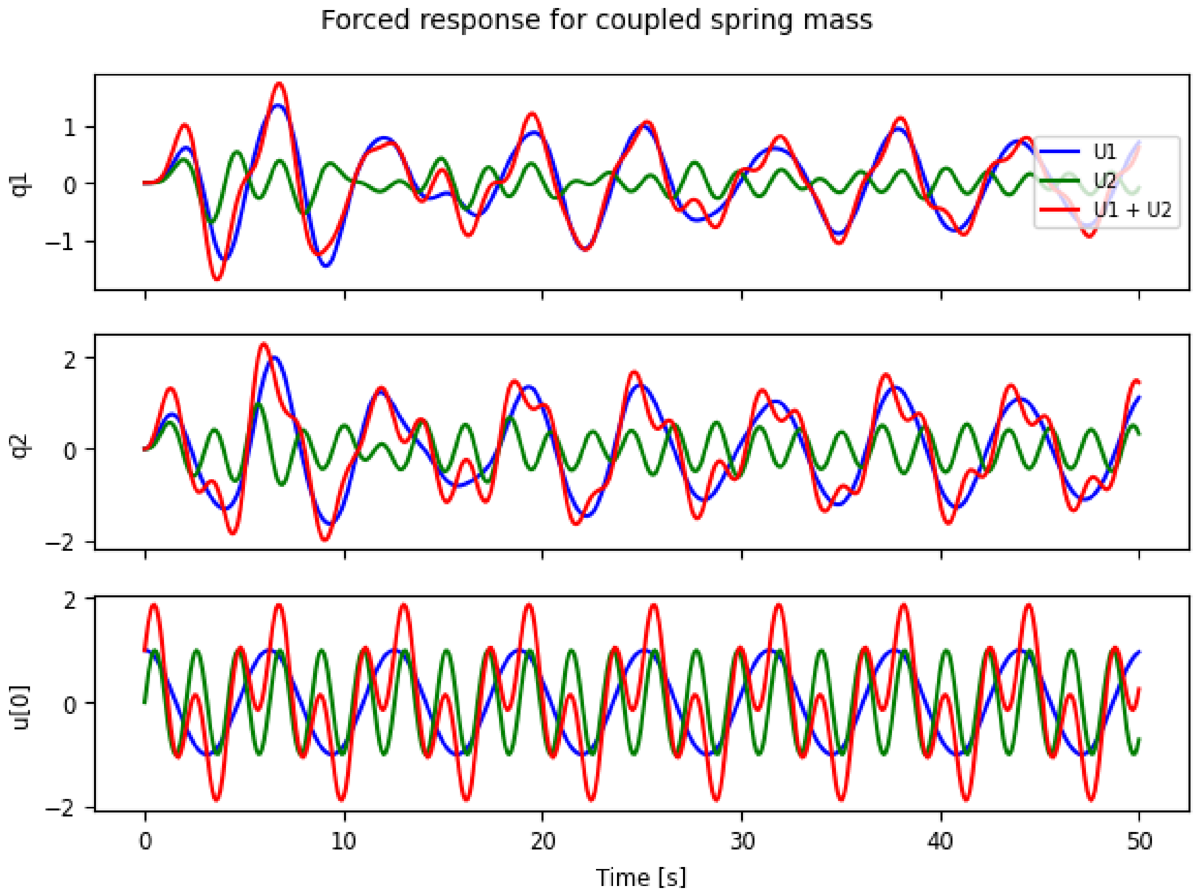
Forced response

To compute the response to an input, using the convolution equation, we can use the `forced_response` function:

In [12]:
```
T = np.linspace(0, 50, 500)
U1 = np.cos(T)
U2 = np.sin(3 * T)

resp1 = ct.forced_response(sys, T, U1)
resp2 = ct.forced_response(sys, T, U2)
resp3 = ct.forced_response(sys, T, U1 + U2)

# Plot the individual responses
resp1.sysname = 'U1'; resp1.plot(color='b')
```

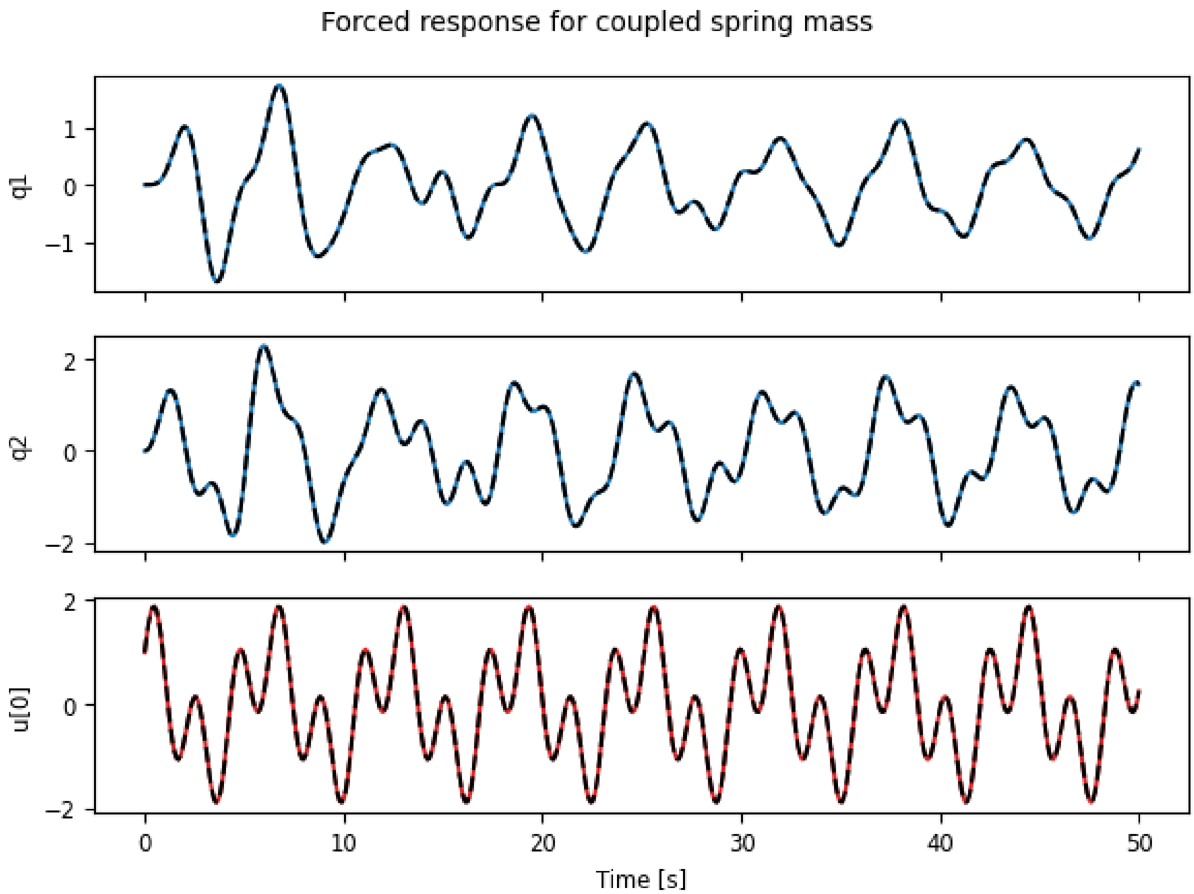
```
resp2.sysname = 'U2'; resp2.plot(color='g')
resp3.sysname = 'U1 + U2'; resp3.plot(color='r')
```

Out[12]:
```
array([[list([<matplotlib.lines.Line2D object at 0x11c0c8ec0>])],
       [list([<matplotlib.lines.Line2D object at 0x11c0c92b0>])],
       [list([<matplotlib.lines.Line2D object at 0x11c0c9130>])]],
      dtype=object)
```



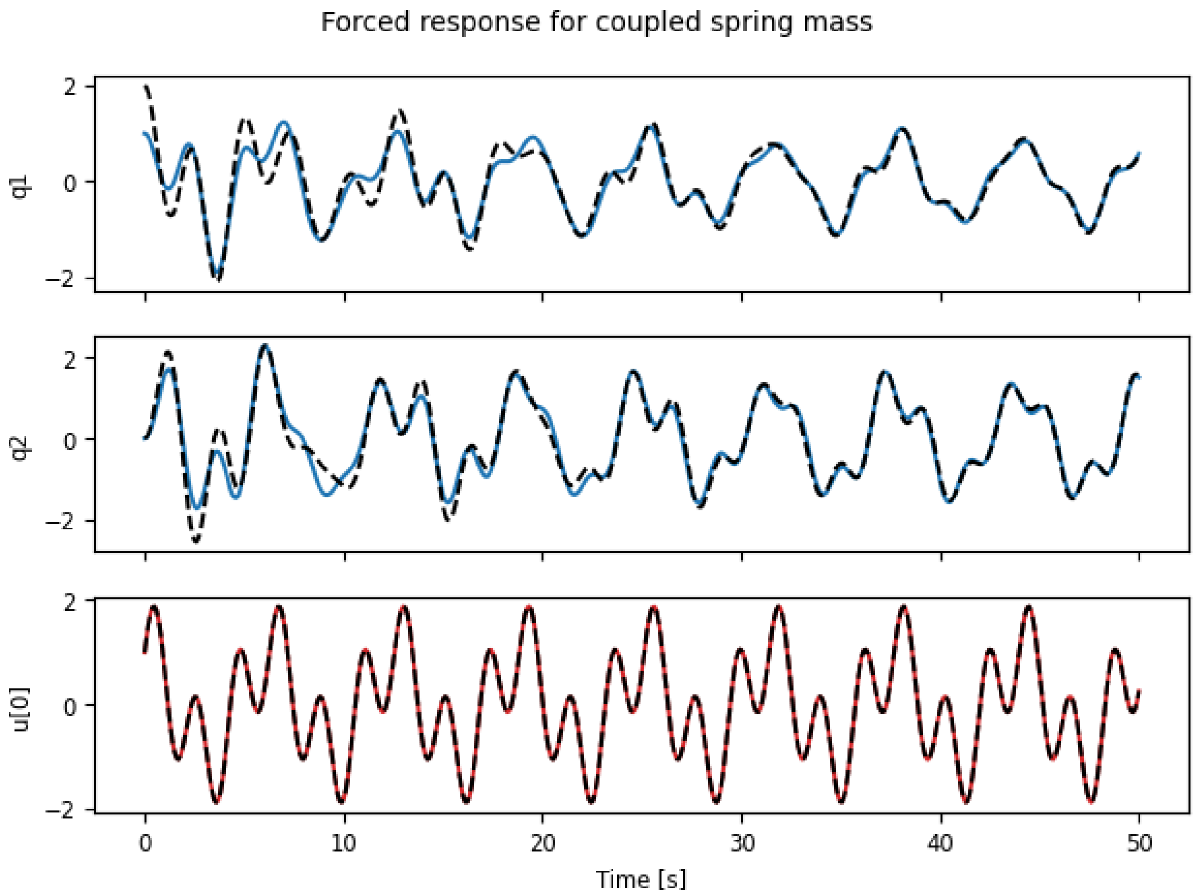
In [13]:
```
# Show that the system response is linear
out = resp3.plot()
axs = ct.get_plot_axes(out)
axs[0, 0].plot(resp1.time, resp1.outputs[0] + resp2.outputs[0], 'k--')
axs[1, 0].plot(resp1.time, resp1.outputs[1] + resp2.outputs[1], 'k--')
axs[2, 0].plot(resp1.time, resp1.inputs[0] + resp2.inputs[0], 'k--')
```

Out[13]:
```
[<matplotlib.lines.Line2D at 0x11c1f4290>]
```



In [14]:
```
# Show that the forced response from non-zero initial condition is not linea
X0 = [1, 0, 0, 0]
resp1 = ct.forced_response(sys, T, U1, X0=X0)
resp2 = ct.forced_response(sys, T, U2, X0=X0)
resp3 = ct.forced_response(sys, T, U1 + U2, X0=X0)

out = resp3.plot()
axs = ct.get_plot_axes(out)
axs[0, 0].plot(resp1.time, resp1.outputs[0] + resp2.outputs[0], 'k--')
axs[1, 0].plot(resp1.time, resp1.outputs[1] + resp2.outputs[1], 'k--')
axs[2, 0].plot(resp1.time, resp1.inputs[0] + resp2.inputs[0], 'k--')
```

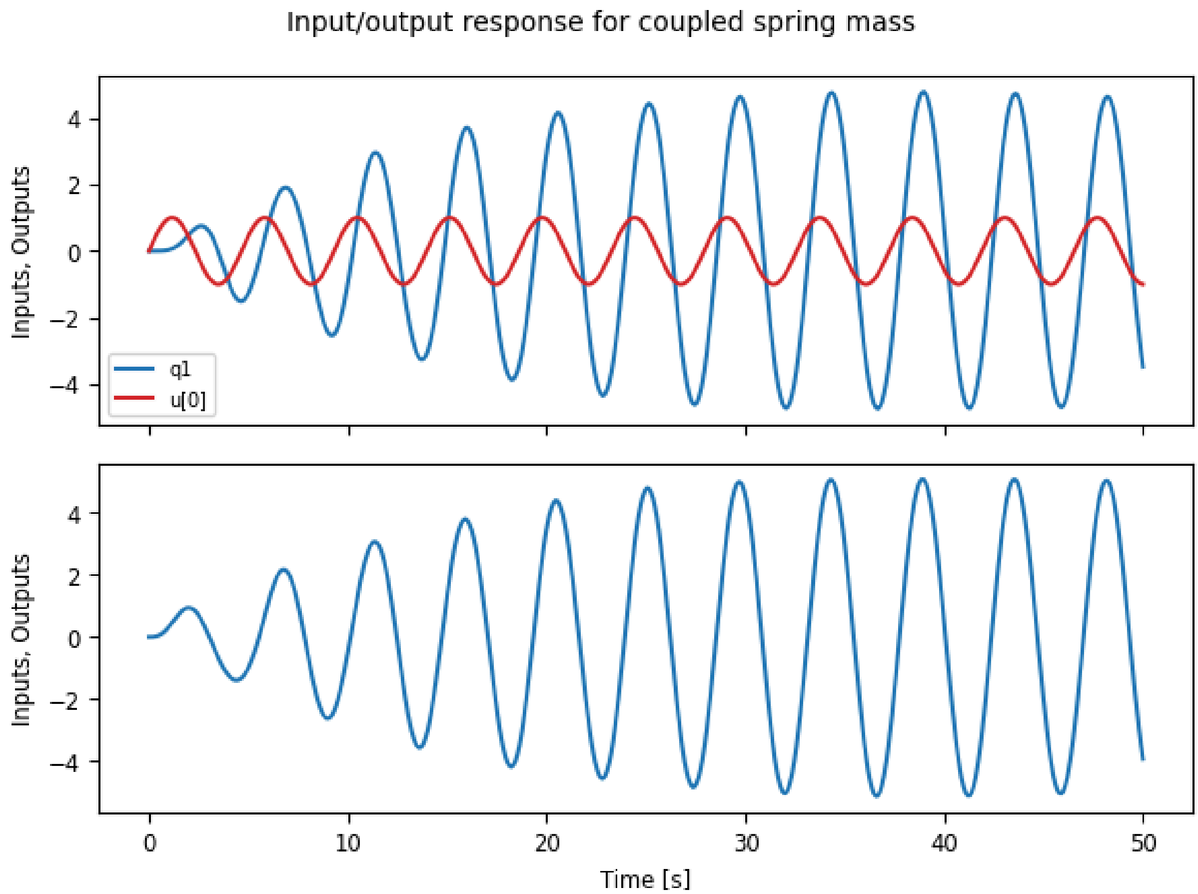
Out[14]:
```
[<matplotlib.lines.Line2D at 0x11c3144a0>]
```



Frequency response

In [15]:
```
# Manual computation of the frequency response
resp = ct.input_output_response(sys, T, np.sin(1.35 * T))

?ct.time_response_plot
resp.plot(plot_inputs='overlay', legend_loc='lower left')
```

Out[15]:
```
array([[list([<matplotlib.lines.Line2D object at 0x11c3d4e30>, <matplotlib.
lines.Line2D object at 0x11c3d5340>])],
       [list([<matplotlib.lines.Line2D object at 0x11c3d5070>])]],
      dtype=object)
```



The magnitude and phase of the frequency response is controlled by the transfer function,

$$G(s) = C(sI - A)^{-1}B + D$$

which can be computed using the `ss2tf` function:

In [16]:
```
# Create SISO transfer functions, since we don't have slycot
G1 = ct.ss2tf(sys[0, 0], name='u to q1')
G2 = ct.ss2tf(sys[1, 0], name='u to q2')
print(G1)
print(G2)
```

```
<TransferFunction>: u to q1
Inputs (1): ['u[0]']
Outputs (1): ['q1']


                  4
-------------------------------------
s^4 + 0.2 s^3 + 8.01 s^2 + 0.8 s + 12

<TransferFunction>: u to q2
Inputs (1): ['u[0]']
Outputs (1): ['q2']


          2 s^2 + 0.2 s + 8
-------------------------------------
s^4 + 0.2 s^3 + 8.01 s^2 + 0.8 s + 12
```

In [17]:

```python
# Gain and phase for the simulation above
from math import pi
val = G1(1.35j)
print(f"{G1(1.35j)=}")
print(f"Gain: {np.absolute(val)}")
print(f"Phase: {np.angle(val)}", " (", np.angle(val) * 180/pi, "deg)")
```

```
G1(1.35j)=(3.3300564744031984-2.706863274436471j)
Gain: 4.291431568743418
Phase: -0.6825322008139448  ( -39.106214488414615 deg)
```

In [18]:

```python
# Gain and phase at s = 0 (= steady state step response)
print(f"{G1(0)=}")
print("Final value of step response:", stepresp.outputs[0, 0, -1])
```

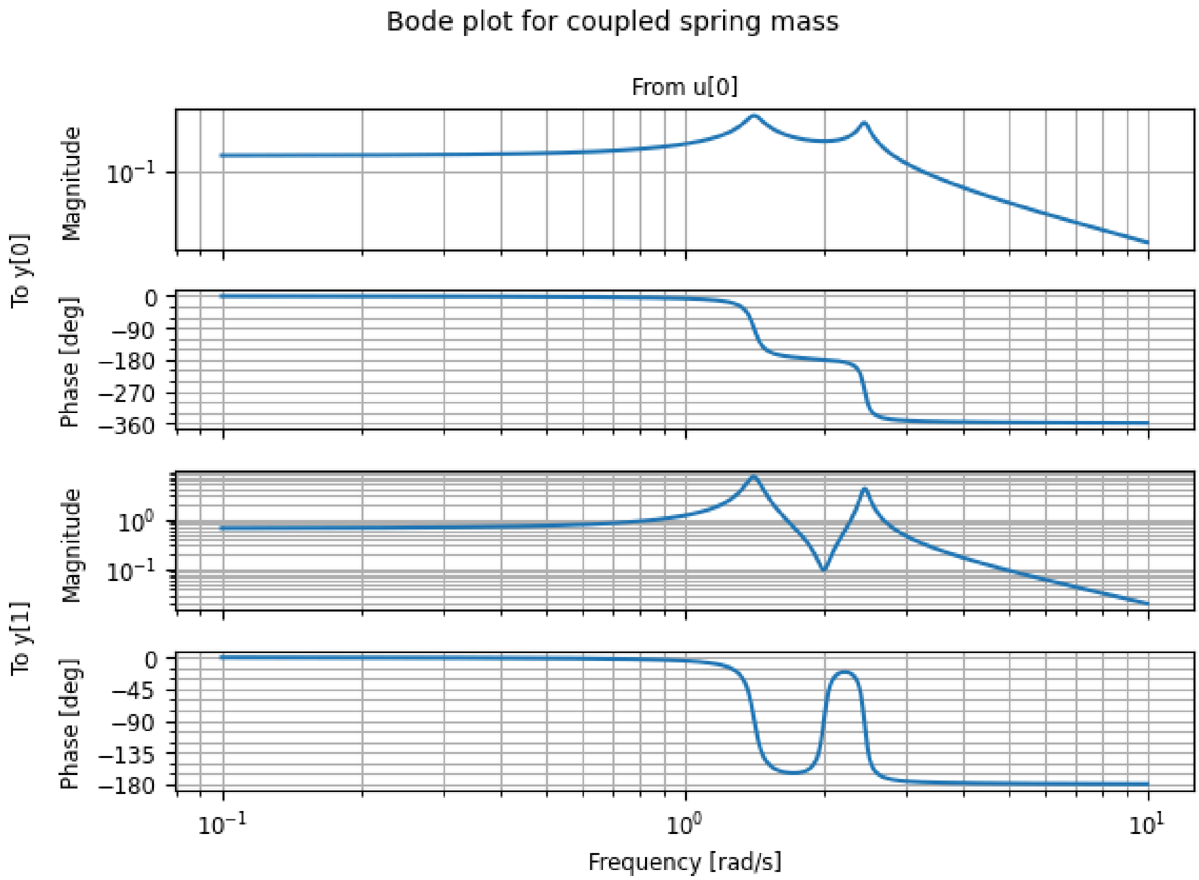
```
G1(0)=(0.3333333333333333+0j)
Final value of step response: 0.33297541813724874
```

In [19]:

```python
ct.bode_plot(sys)
```

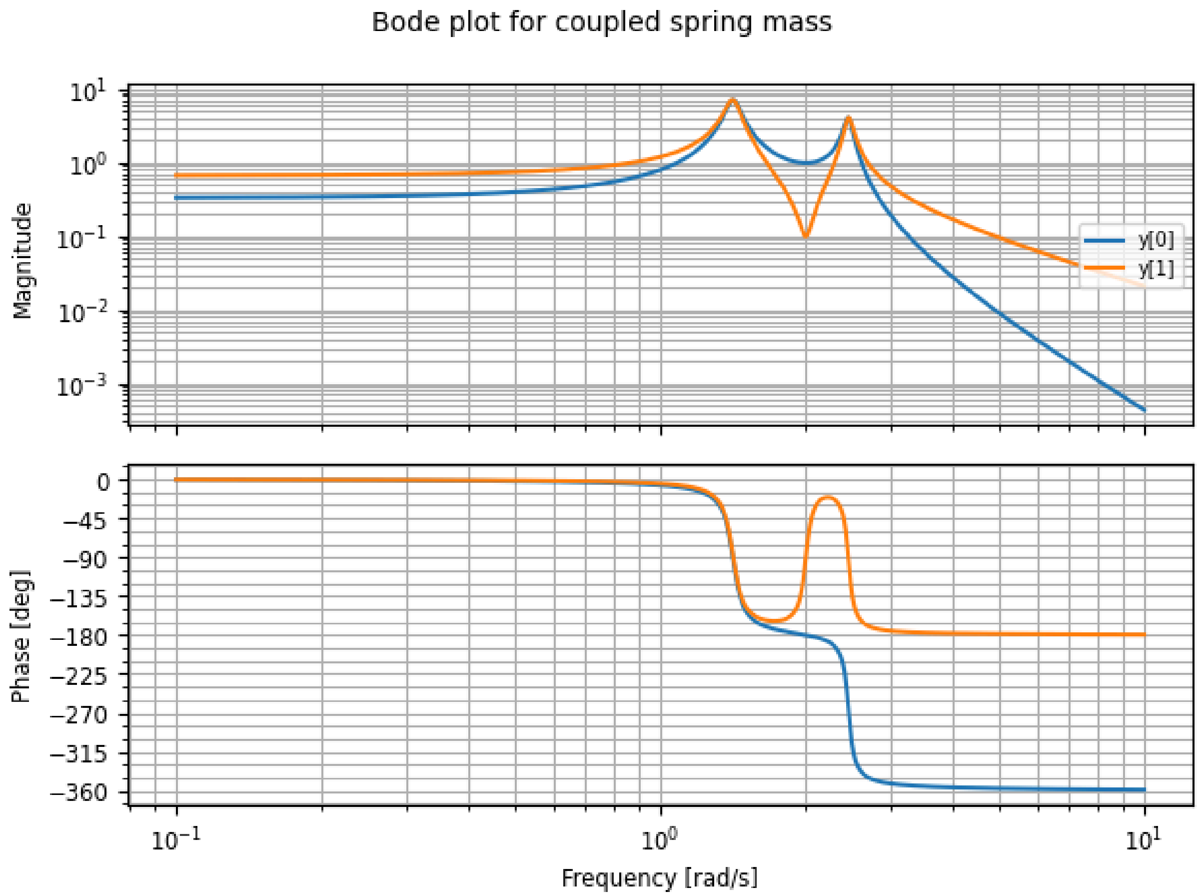
Out[19]:

```
array([[list([<matplotlib.lines.Line2D object at 0x11c203b90>])],
       [list([<matplotlib.lines.Line2D object at 0x11bfc2000>])],
       [list([<matplotlib.lines.Line2D object at 0x11bfc20c0>])],
       [list([<matplotlib.lines.Line2D object at 0x11c139610>])]],
      dtype=object)
```



In [20]:
```
ct.bode_plot(sys, overlay_outputs=True)
```

Out[20]:
```
array([[list([<matplotlib.lines.Line2D object at 0x11c83e600>, <matplotlib.
lines.Line2D object at 0x11c902780>])],
       [list([<matplotlib.lines.Line2D object at 0x11c902390>, <matplotlib.
lines.Line2D object at 0x11c902a20>])]],
      dtype=object)
```

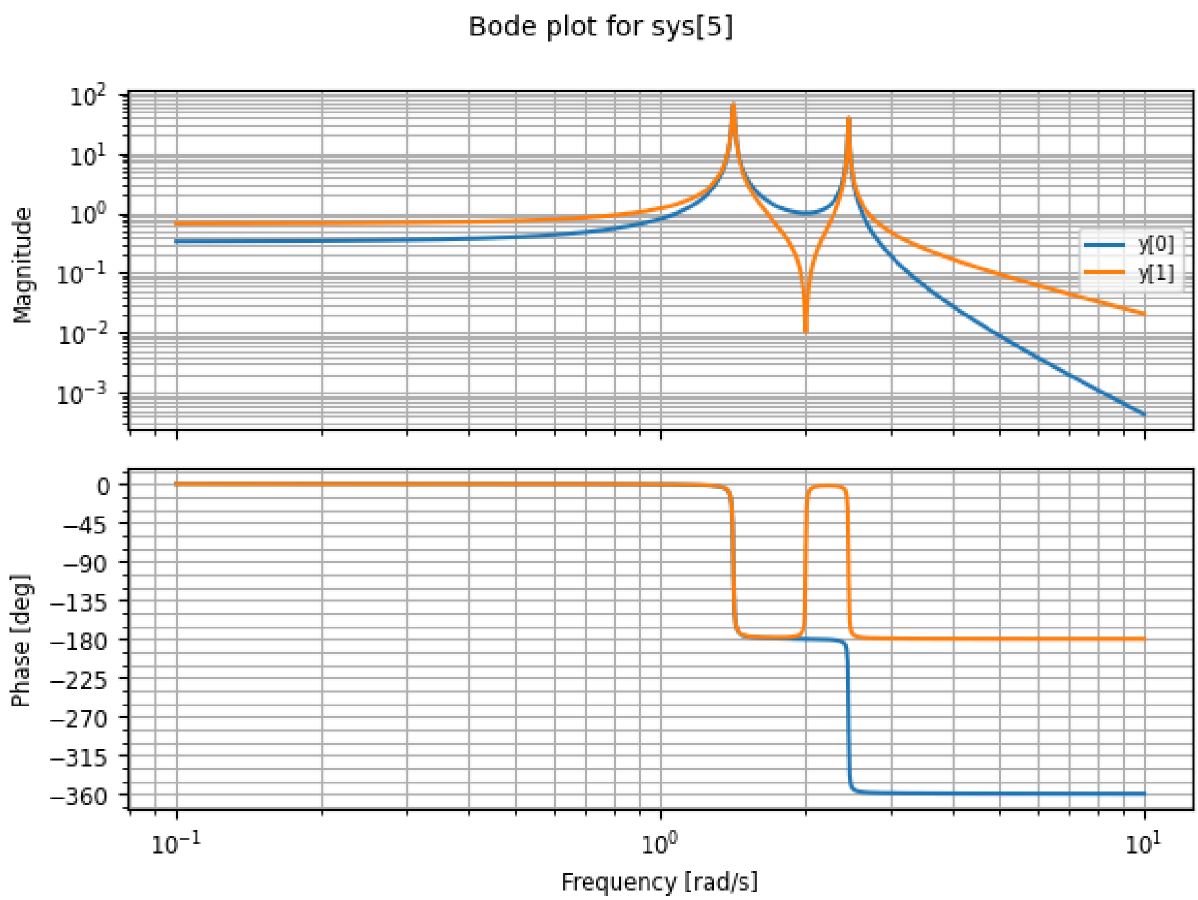


Note the "dip" in the frequency response for y[1] at frequency 2 rad/sec, which corresponds to a "zero" of the transfer function.

This dip becomes even more pronounced in the case of low damping coefficient c:

In [21]:
```
ct.bode_plot(
    coupled.linearize([0, 0, 0, 0], [0], params={'c': 0.01}),
    overlay_outputs=True)
```

Out[21]:
```
array([[list([<matplotlib.lines.Line2D object at 0x11cbd3650>, <matplotlib.
lines.Line2D object at 0x11cbd3d10>])],
       [list([<matplotlib.lines.Line2D object at 0x11cbd3980>, <matplotlib.
lines.Line2D object at 0x11cbd3fe0>])]],
      dtype=object)
```



Additional resources

- Code for FBS2e figures: Python code used to generate figures in FBS2e
- Python-control documentation for plotting time responses
- Python-control documentation for plotting frequency responses
- Python-control examples: lots of Python and Jupyter examples of control system analysis and design

In [ ]: