$\boxed{\text{CDS 110, Lecture 3.2: Linear System Response}}$ RMM 15 Apr 2024

I. Convolution equation

$\dot{x} = Ax + Bu \quad x(t_0)$ given
$y = Cx + Du$

$$x(t) = \underbrace{e^{A(t-t_0)}x(t_0)}_{\text{initial cond resp}} + \underbrace{\int_{t_0}^{t} \underbrace{e^{A(t-\tau)}B}_{\text{impulse response}} u(\tau)\,d\tau}_{\text{forced response}}$$

$$y(t) = Cx(t) + Du(t) = \cdots$$

Pf: substitution: $\dot{x}(t) = Ae^{A(t-t_0)}x(t_0) + \frac{d}{dt}\Big[\underbrace{e^{At}}_{\alpha(t)} \underbrace{\int_{t_0}^{t} e^{-A\tau}Bu(\tau)\,d\tau}_{\beta(t)}\Big]$

$$= Ae^{A(t-t_0)}x(t_0) + \alpha'(t)\beta(t) + \alpha(t)\beta'(t)$$
$$= \quad '' \quad + Ae^{At}\int_{t_0}^{t} \cdots d\tau + e^{At}e^{-At}Bu(t)$$
$$= A\underbrace{\Big[e^{A(t-t_0)}x(t_0) + e^{At}\int_{t_0}^{t}\cdots d\tau\Big]}_{x(t)} + Bu(t) \quad \checkmark$$

II. Stability: $z = Tx \quad \dot{z} = T\dot{x} = \underbrace{TAT^{-1}}_{\tilde{A}}x + TBu$

$e^{\tilde{A}t} = Te^{At}T^{-1}$ (pf: exercise) $\Rightarrow x(t) = T^{-1}z(t) = T^{-1}\left(e^{\tilde{A}t}x(0) + \cdots\right)$

Suppose evals unique $\Rightarrow$ can diagonalize $\Rightarrow \tilde{A} = \begin{bmatrix}\lambda_1 & & \\ & \ddots & \\ & & \lambda_n\end{bmatrix} \Rightarrow e^{\tilde{A}t} = \begin{bmatrix}e^{\lambda_1 t} & & \\ & \ddots & \\ & & e^{\lambda_n t}\end{bmatrix}$

<u>Thm</u> For any $A \in \mathbb{R}^{n\times n}$, $\exists\, T$ s.t. TAT^{-1} in Jordan form $J = \begin{bmatrix}J_1 & & \\ & \ddots & \\ & & J_k\end{bmatrix}$ $J_k = \begin{bmatrix}\lambda_k & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & \lambda_k\end{bmatrix}$

$$e^{J_k t} = \begin{bmatrix} 1 & t & t^2/2 & \cdots & \frac{t^{n-1}}{(n-1)!} \\ & \ddots & \ddots & \ddots & \vdots \\ & & \ddots & \ddots & \\ & 0 & & \ddots & t \\ & & & & 1 \end{bmatrix} e^{\lambda_k t}$$

<u>Thm</u> asy stable $Re(\lambda) < 0$
unstable $Re(\lambda_i) > 0$
$Re(\lambda_i) = 0 \Rightarrow$ depends on J_k

III. Exponential response: $e^{i\omega t} = \cos\omega t + i\sin\omega t$)

$\cos\omega t = \frac{1}{2}\left(e^{i\omega t} + e^{-i\omega t}\right) \Rightarrow$ compute response for e^{st}, $s = i\omega$

$$x(t) = e^{At}x(0) + \int_0^t e^{A(t-\tau)}Be^{s\tau}\,d\tau = \quad '' \quad + e^{At}\int_0^t e^{(sI-A)\tau}B\,d\tau$$
$$= \quad '' \quad + e^{At}\Big[(sI-A)^{-1}e^{(sI-A)\tau}B\Big]\Big|_{\tau=0}^{\tau=t}$$
$$= \quad '' \quad + e^{At}(sI-A)^{-1}\left[e^{(sI-A)t} - I\right]B$$

$$= e^{At}x(0) - e^{At}(sI-A)^{-1}B + (sI-A)^{-1}Be^{st}$$

$$y(t) = \underbrace{Ce^{At}x(0) - Ce^{At}(sI-A)^{-1}B}_{\text{transient } \to 0} + \underbrace{\left[C(sI-A)^{-1}B + D\right]}_{\text{steady state: } G(s)e^{st}}e^{st}$$