CDS 110/ChE 105: Lecture 3.1
Linear Systems

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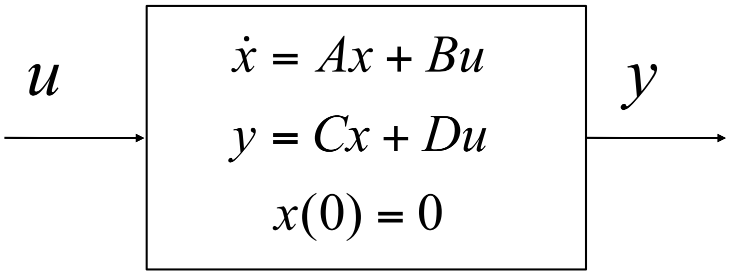
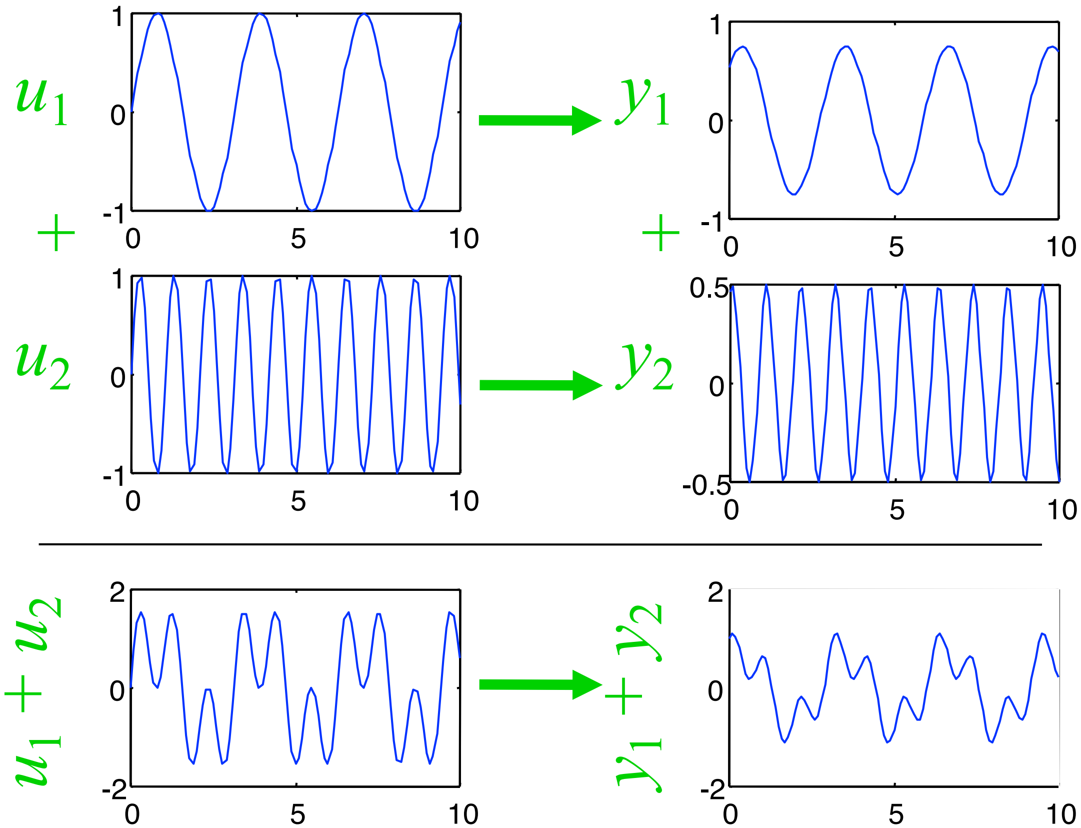
Goals:
- Describe linear system models: properties, examples, and tools
- Characterize (stability &) performance of linear systems in terms of eigenvalues
- Compute linearization of a nonlinear systems around an equilibrium point

Reading:
- Åström and Murray, *Analysis and Design of Feedback Systems,* Ch 6

Linear Systems





Input/output linearity at x(0) = 0

- Linear systems are linear in initial condition *and* input ⇒ need to use x(0) = 0 to add outputs together
- For different initial conditions, you need to be more careful

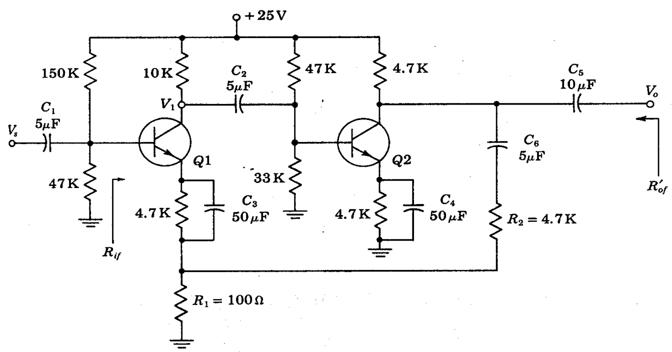
Linear system ⇒ step response and frequency response scale with input amplitude

- 2X input ⇒ 2X output
- Allows us to use *ratios* and *percentages* in step/freq response. These are *independent* of input amplitude
- Limitation: input saturation ⇒ only holds up to certain input amplitude

Why are Linear Systems Important?

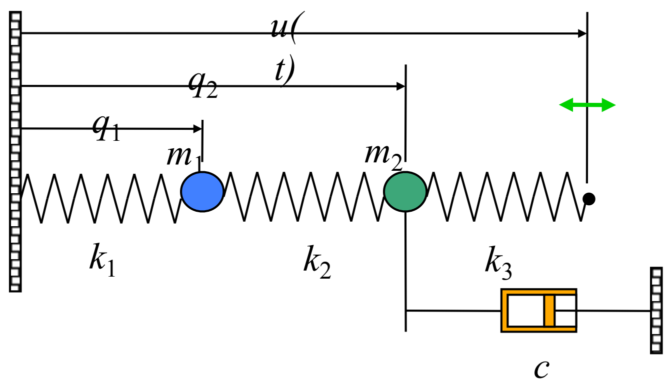
Many important *examples*

Electronic circuits



- Especially true after feedback
- Frequency response is key performance specification

Many mechanical systems



Almost anything near equil pts

Many important *tools*

Frequency response, step response, etc

- Traditional tools of control theory
- Developed in 1930's at Bell Labs; intercontinental telecom

Classical control design toolbox } CDS 110/221

- Nyquist plots, gain/phase margin
- Loop shaping

Optimal control and estimators } CDS 110/212

- Linear quadratic regulators
- Kalman estimators

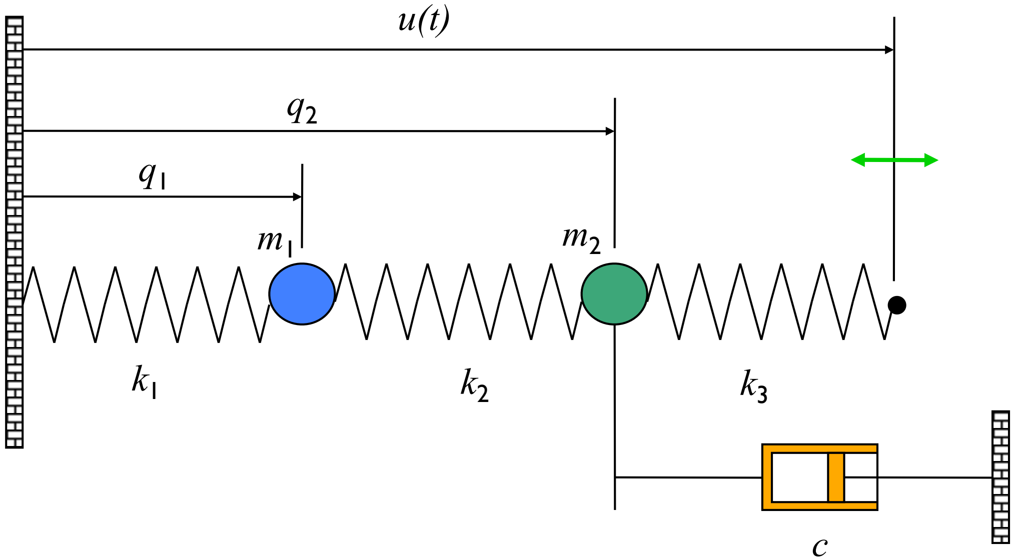
Robust control design } CDS 131/221

- H_1 control design
- μ analysis for structured uncertainty

Serves as basis for most nonlin methods

Not included in handout

Example #1: Spring Mass System



Applications

- Flexible structures (many apps)
- Suspension systems (eg, cars)
- Molecular and quantum dynamics

Questions we want to answer

- How much do masses move as a function of the forcing frequency?
- What happens if I change the values of the masses?
- Will the car fly into the air if I take that speed bump at 25 mph?

Modeling assumptions

- Mass, spring, and damper constants are fixed and known
- Springs satisfy Hooke's law
- Damper is (linear) viscous force, proportional to velocity

Solutions of Linear Systems: The Matrix Exponential

$$\dot{x} = Ax + Bu \qquad \longrightarrow \qquad y(t) = ???$$
$$y = Cx + Du$$

Scalar linear system, with no input

$$\dot{x} = ax \qquad x(0) = x_0 \longrightarrow x(t) = e^{at}x_0 \longrightarrow y(t) = ce^{at}x_0$$
$$y = cx$$

Matrix version, with no input

$$\dot{x} = Ax \qquad x(0) = x_0 \longrightarrow x(t) = e^{At}x_0 \longrightarrow \boxed{y(t) = Ce^{At}x_0}$$
$$y = Cx$$

initial_response(sys, x0)

Matrix exponential

- Analog to the scalar case; defined by series expansion:

$$e^{M} = I + M + \frac{1}{2!}M^2 + \frac{1}{3!}M^3 + \cdots$$

S = np.linalg.expm(M)

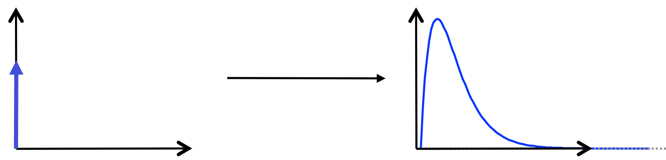
Linear Control Systems and Convolution

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

$$\longrightarrow \quad y(t) = \underbrace{Ce^{At}x(0)}_{\text{homogeneous}} + ???$$

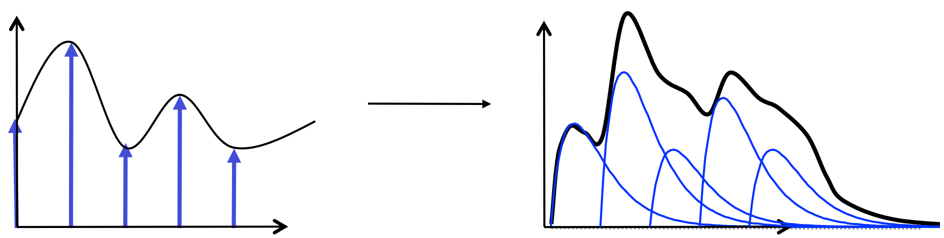
Impulse response, h(t) = Ce^At^B — $h(t) = Ce^{At}B$

- Response to input “impulse”
- Equivalent to “Green’s function”



Linearity ⇒ compose response to arbitrary *u*(*t*) using *convolution*

- Decompose input into “sum” of shifted impulse functions
- Compute impulse response for each
- “Sum” impulse response to find *y*(*t*)



Complete solution: use integral instead of “sum”

$$y(t) = Ce^{At}x(0) + \int_{\tau=0}^{t} Ce^{A(t-\tau)}Bu(\tau)\,d\tau + Du(t)$$

- linear with respect to initial condition *and* input
- 2X input ⇒ 2X output when $x(0) = 0$

Input/Output Performance

Return to system with inputs

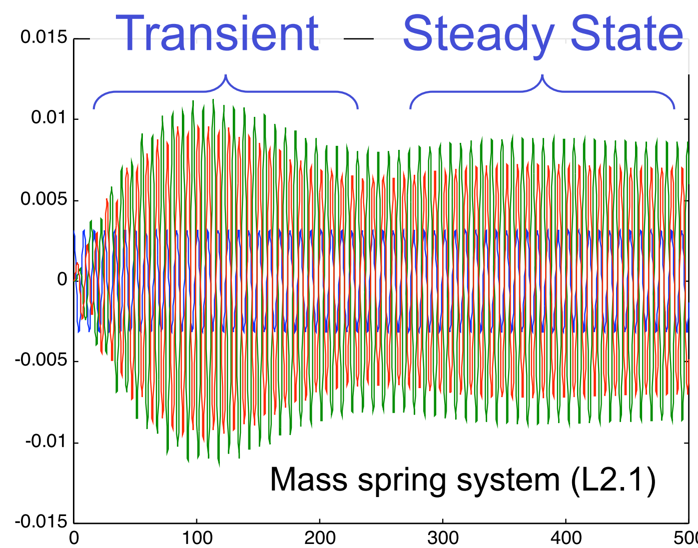
- How does system response to changes in input values?

Transient response:

- What happens right after a new input is applied

Steady state response:

- What happens a long time after the input is applied



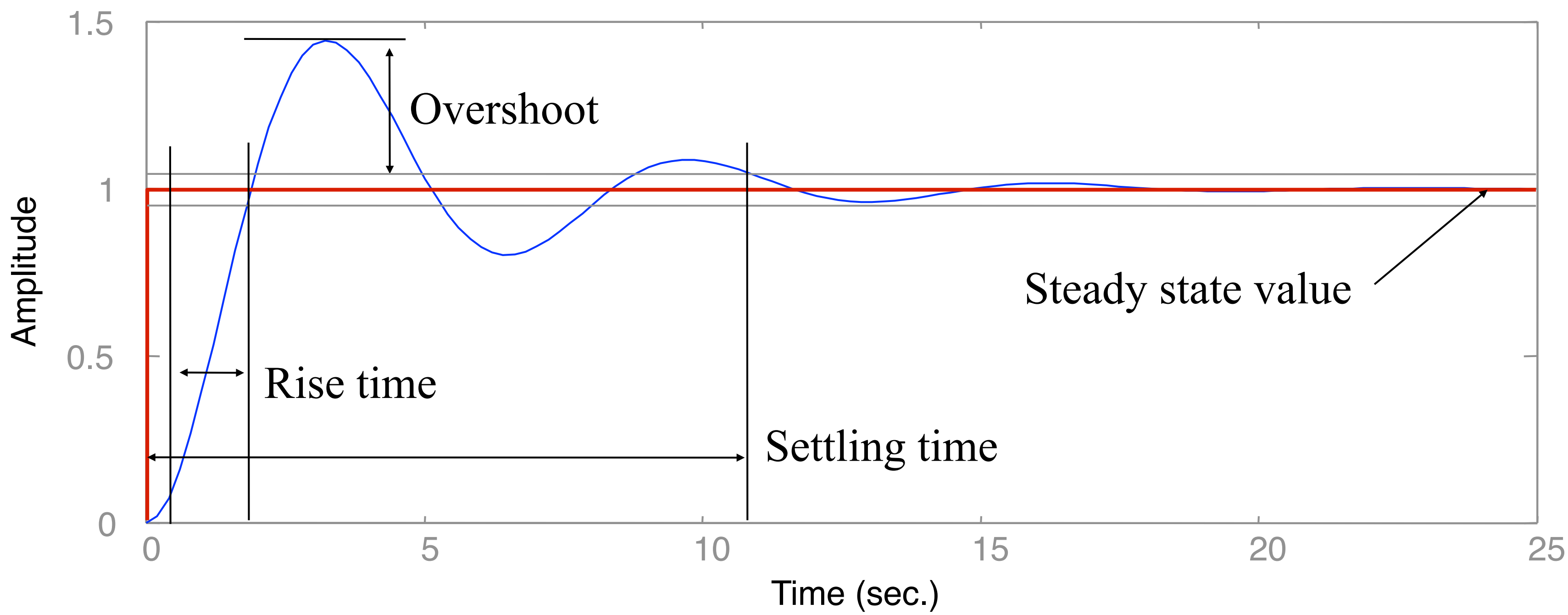
Stability vs input/output performance

- Systems that are close to instability typically exhibit poor input/output performan
- Nearly unstable systems (slow convergence) often exhibit “ringing” (highly oscillatory response to [non-periodic] inputs)

Step Response

Output characteristics in response to a "step" input

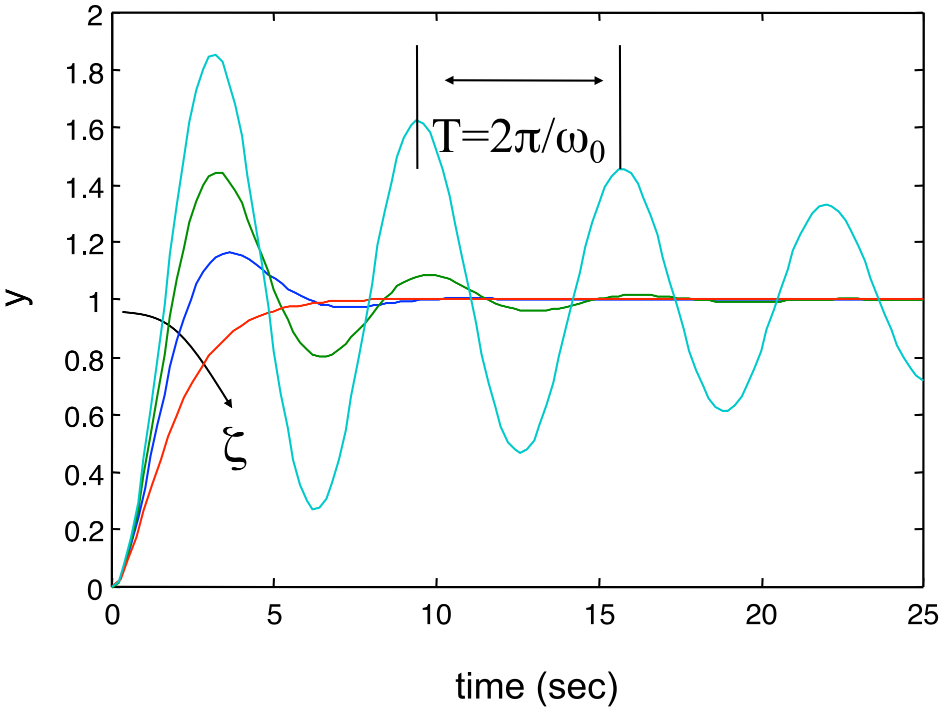
- Rise time: time required to move from 5% to 95% of final value
- Overshoot: ratio between amplitude of first peak and steady state value
- Settling time: time required to remain w/in *p*% (usually 2%) of final value
- Steady state value: final value at $t = \infty$

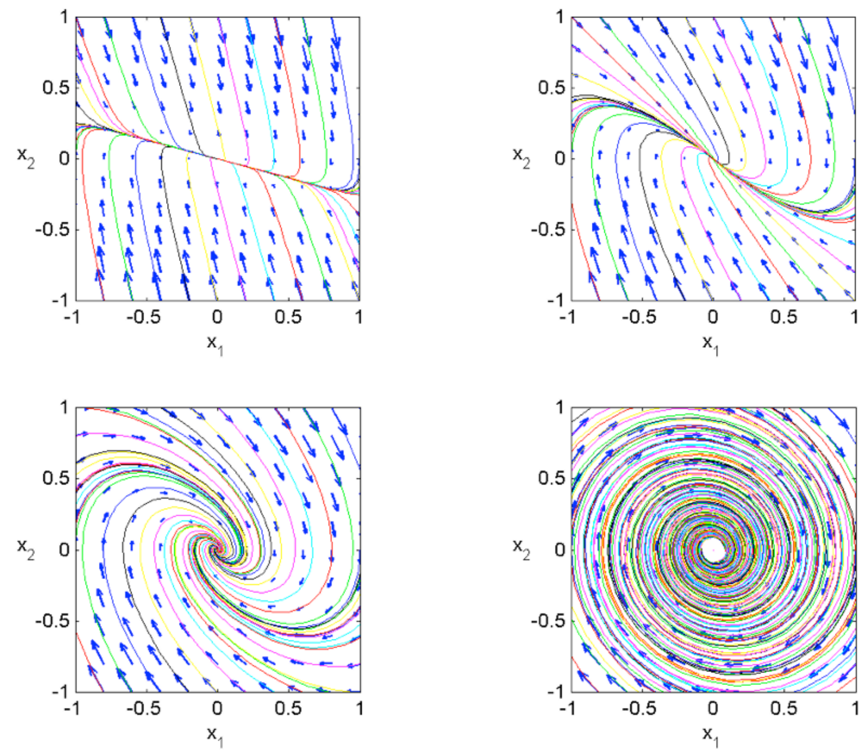


Second Order Systems

Important class of systems in many applications areas

$$\ddot{q} + 2\zeta\omega_0\dot{q} + \omega_0^2 q = u \quad \longleftrightarrow \quad \frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\omega_0^2 & -2\zeta\omega_0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$



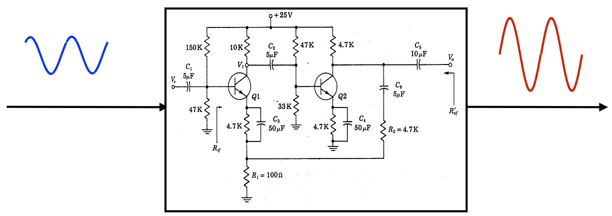


- Analytical formulas exist for overshoot, rise time, settling time, etc

Frequency Response

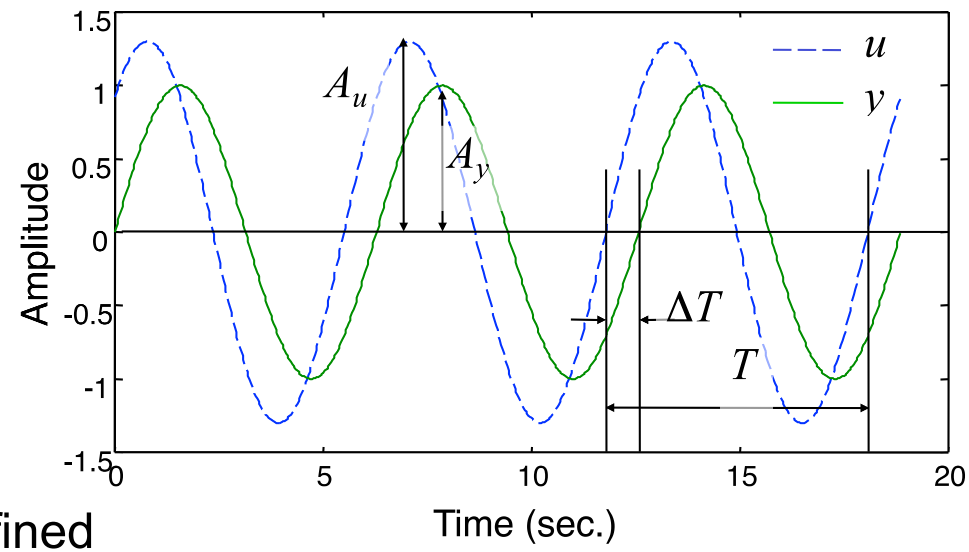
Measure the *steady state* response of the system to sinusoidal input

- Example: audio amplifier – would like consistent ("flat") amplification between 20 Hz & 20,000 Hz
- Individual sinusoids are good *test signals* for measuring performance in many systems (eg, seasonal cycles in temperature)



Approach: plot input and output, measure *relative* amplitude and phase

- Generate response of system to sinusoidal output
- Gain = A_y/A_u
- Phase = $2\pi \cdot \Delta T/T$



May not work for *nonlinear* systems

- System nonlinearities can cause *harmonics* to appear in the output
- Amplitude and phase may not be well-defined
- For *linear* systems, frequency response is always well defined

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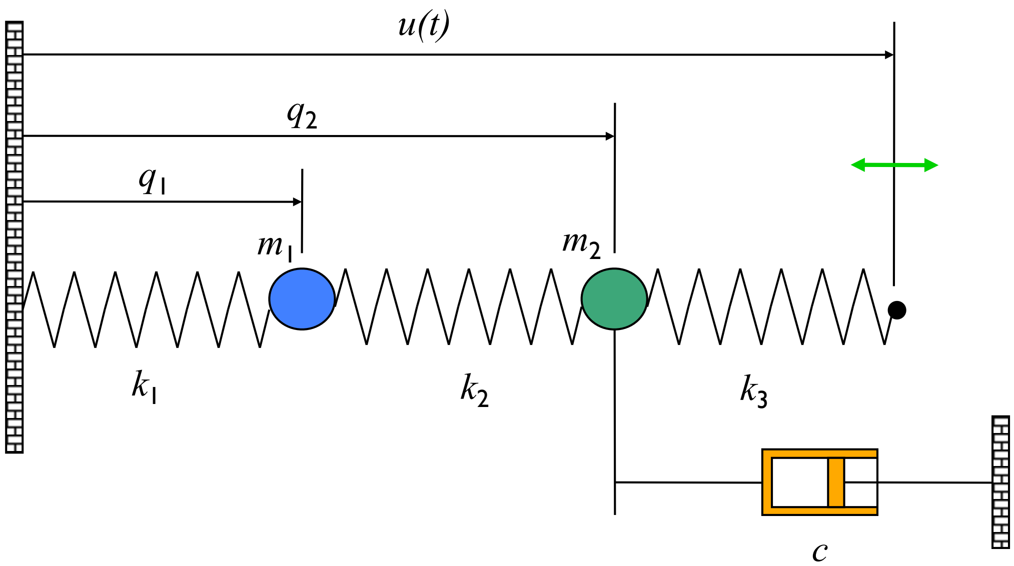

Applications

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Questions we want to answer

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Modeling assumptions

- Mass, spring, and damper constants are fixed and known
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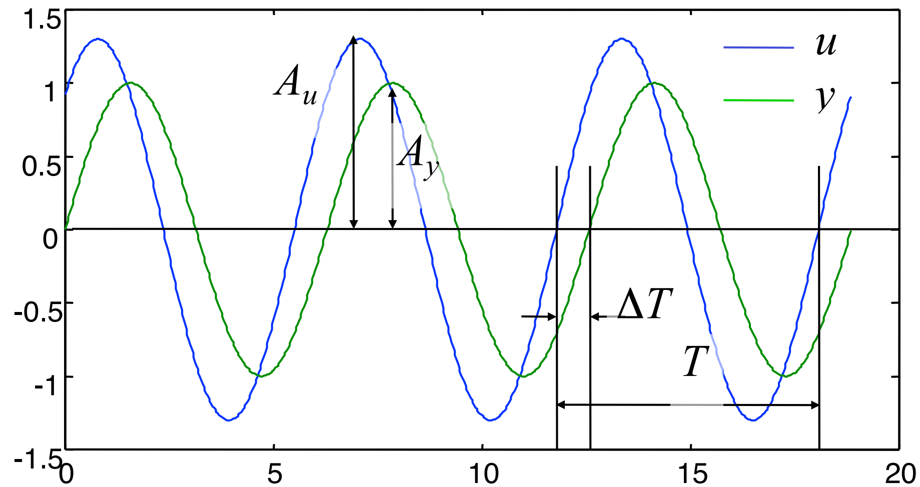
Computing Frequency Responses

Technique #1: plot input and output, measure relative amplitude and phase

- Use input_output_response to generate response of system to sinusoidal output
- Gain = A_y/A_u
- Phase = $2\pi \cdot \Delta T/T$
- Note: In general, gain and phase will depend on the input amplitude



Technique #2 (linear systems): use Python bode command

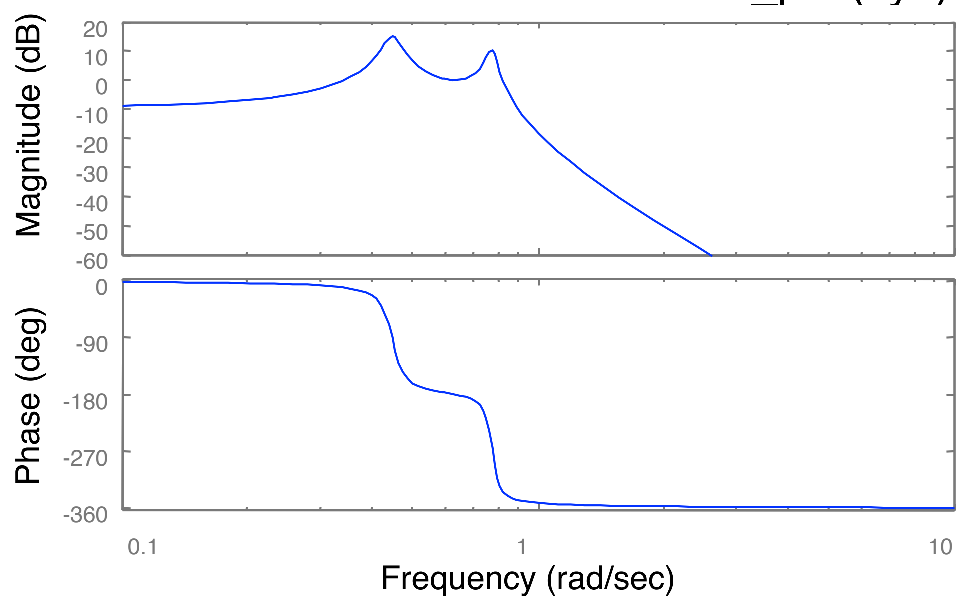
- Assumes linear dynamics in state space form:

$$\dot{x} = Ax + Bu$$

$$y = Cx + Du$$

- Gain plotted on log-log scale
  - Traditional: dB = 20 $\log_{10}$ (gain)
- Phase plotted on linear-log scale

ct.bode_plot(sys)



Linearization Around an Equilibrium Point

$$\dot{x} = f(x,u) \qquad \longrightarrow \qquad \dot{z} = Az + Bv$$
$$y = h(x,u) \qquad\qquad\qquad w = Cz + Dv$$

"Linearize" around $x = x_e$

$$f(x_e, u_e) = 0 \qquad y_e = h(x_e, u_e)$$

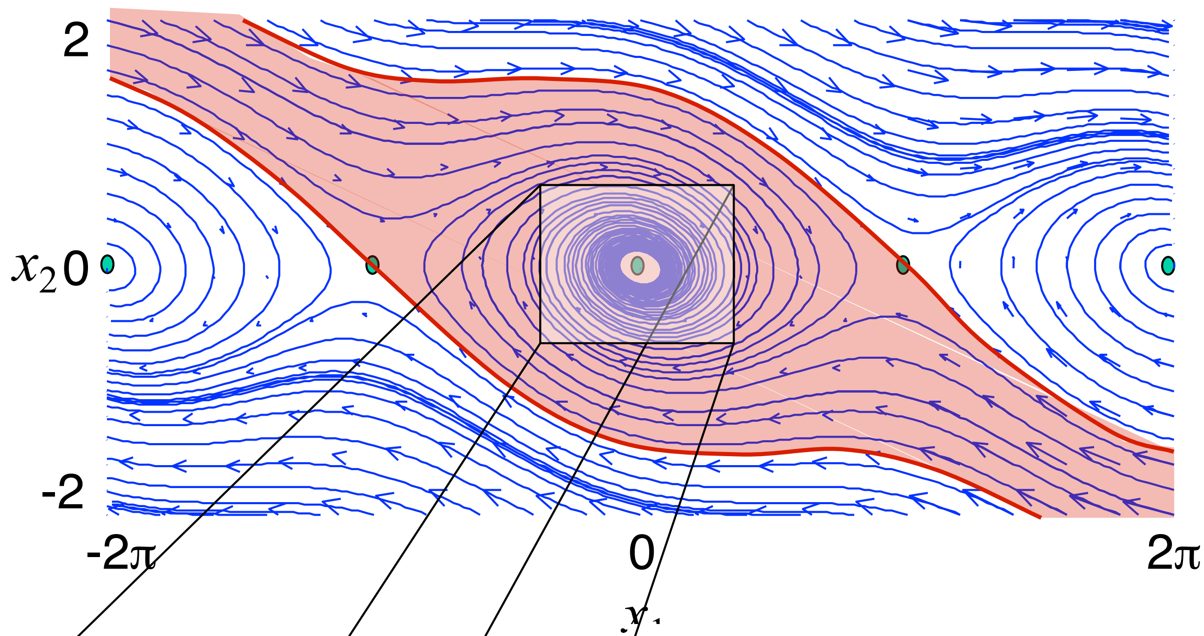
$$z = x - x_e \qquad v = u - u_e \qquad w = y - y_e$$

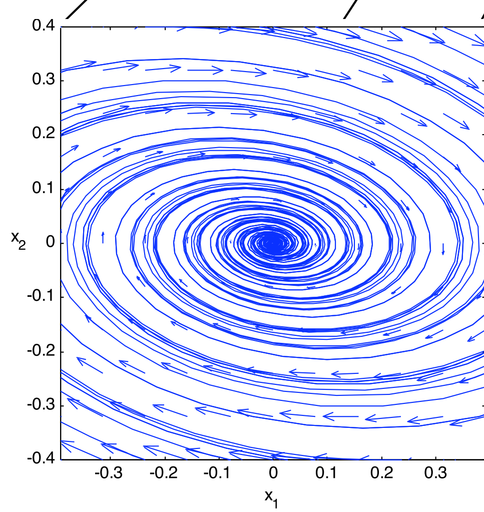
$$A = \left.\frac{\partial f}{\partial x}\right|_{(x_e,u_e)} \qquad B = \left.\frac{\partial f}{\partial u}\right|_{(x_e,u_e)}$$

$$C = \left.\frac{\partial h}{\partial x}\right|_{(x_e,u_e)} \qquad D = \left.\frac{\partial h}{\partial u}\right|_{(x_e,u_e)}$$

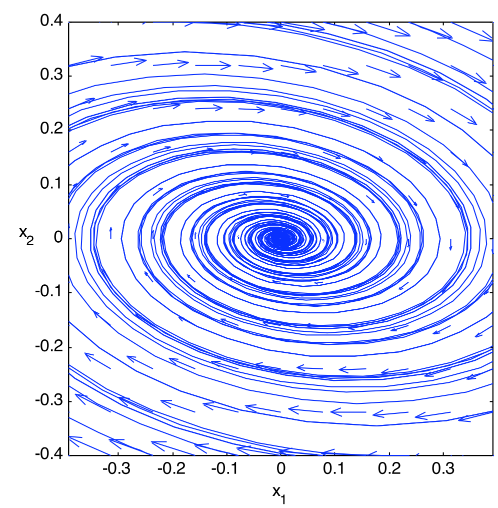
Remarks

- In examples, this is often equivalent to small angle approximations, etc
- Only works *near* to equilibrium point





Full nonlinear model



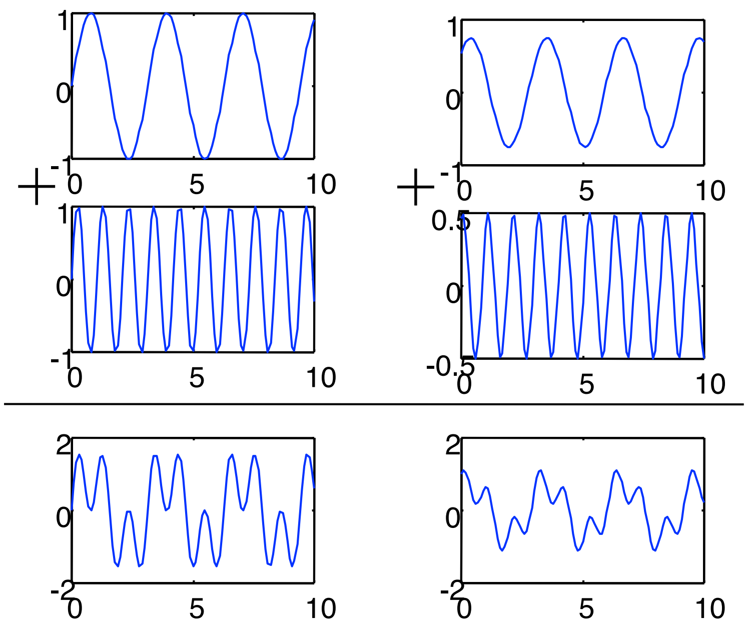
Linear model (honest!)

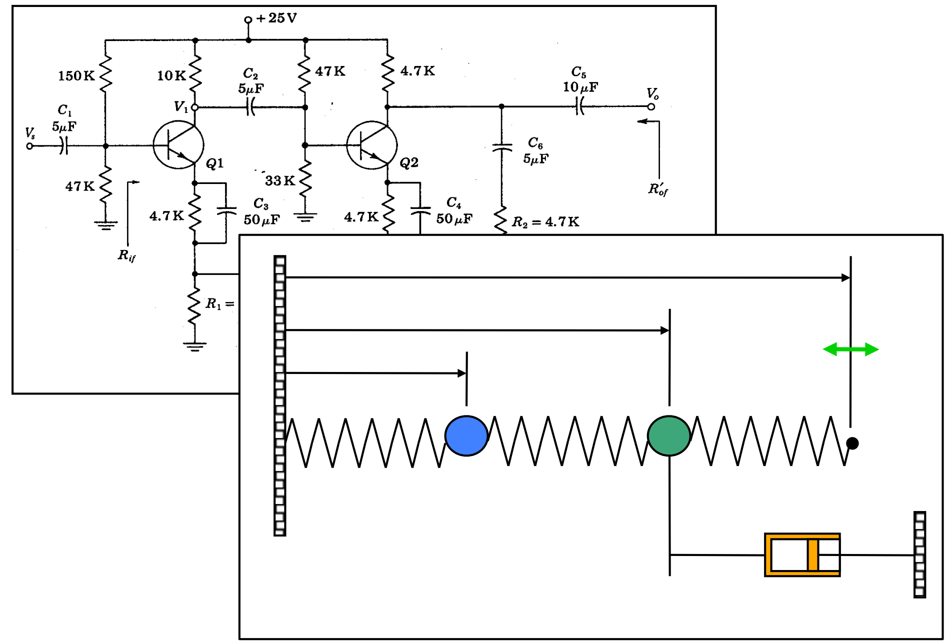
Summary: Linear Systems

u →

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$
$$x(0) = 0$$

→ y





$$y(t) = Ce^{At}x(0) + \int_{\tau=0}^{t} Ce^{A(t-\tau)}Bu(\tau)d\tau + Du(t)$$

Properties of linear systems

- Linearity with respect to initial condition and inputs
- Stability characterized by eigenvalues
- Many applications and tools available
- Provide local description for nonlinear systems