Goals:

- Describe linear system models: properties, examples, and tools
- Characterize (stability &) performance of linear systems in terms of eigenvalues
- Compute linearization of a nonlinear systems around an equilibrium point

Reading:

- Åström and Murray, *Analysis and Design of Feedback Systems*, Ch 6
Input/output linearity at $x(0) = 0$

- Linear systems are linear in initial condition and input $\Rightarrow$ need to use $x(0) = 0$ to add outputs together
- For different initial conditions, you need to be more careful

Linear system $\Rightarrow$ step response and frequency response scale with input amplitude

- 2X input $\Rightarrow$ 2X output
- Allows us to use ratios and percentages in step/freq response. These are independent of input amplitude
- Limitation: input saturation $\Rightarrow$ only holds up to certain input amplitude
Why are Linear Systems Important?

Many important examples

Electronic circuits

- Especially true after feedback
- Frequency response is key performance specification

Many mechanical systems

Almost anything near equil pts

Many important tools

Frequency response, step response, etc
- Traditional tools of control theory
- Developed in 1930’s at Bell Labs; intercontinental telecom

Classical control design toolbox
- Nyquist plots, gain/phase margin
- Loop shaping

Optimal control and estimators
- Linear quadratic regulators
- Kalman estimators

Robust control design
- $H_1$ control design
- $\mu$ analysis for structured uncertainty

Serves as basis for most nonlin methods
Example #1: Spring Mass System

Applications
- Flexible structures (many apps)
- Suspension systems (eg, cars)
- Molecular and quantum dynamics

Questions we want to answer
- How much do masses move as a function of the forcing frequency?
- What happens if I change the values of the masses?
- Will the car fly into the air if I take that speed bump at 25 mph?

Modeling assumptions
- Mass, spring, and damper constants are fixed and known
- Springs satisfy Hooke’s law
- Damper is (linear) viscous force, proportional to velocity
Solutions of Linear Systems: The Matrix Exponential

\[
\begin{align*}
\dot{x} &= Ax + Bu \\
y &= Cx + Du
\end{align*}
\]

\[y(t) = ???
\]

Scalar linear system, with no input

\[
\begin{align*}
\dot{x} &= ax \\
x(0) &= x_0 \\
y &= cx
\end{align*}
\]

\[x(t) = e^{at} x_0 \quad \rightarrow \quad y(t) = ce^{at} x_0
\]

Matrix version, with no input

\[
\begin{align*}
\dot{x} &= Ax \\
x(0) &= x_0 \\
y &= Cx
\end{align*}
\]

\[x(t) = e^{At} x_0 \quad \rightarrow \quad y(t) = Ce^{At} x_0
\]

\[\text{initial_response(sys, x0)}
\]

Matrix exponential

- Analog to the scalar case; defined by series expansion:

\[
e^M = I + M + \frac{1}{2!} M^2 + \frac{1}{3!} M^3 + \cdots
\]

\[S = \text{np.linalg.expm}(M)
\]
Linear Control Systems and Convolution

\[ \dot{x} = Ax + Bu \]
\[ y = Cx + Du \]
\[ y(t) = Ce^{At}x(0) + ??? \]

**Impulse response,** \( h(t) = Ce^{At}B \)

- Response to input “impulse”
- Equivalent to “Green’s function”

**Linearity ⇒ compose response to arbitrary** \( u(t) \) **using convolution**

- Decompose input into “sum” of shifted impulse functions
- Compute impulse response for each
- “Sum” impulse response to find \( y(t) \)

**Complete solution:** use integral instead of “sum”

\[ y(t) = Ce^{At}x(0) + \int_{\tau=0}^{t} Ce^{A(t-\tau)} Bu(\tau) d\tau + Du(t) \]

- linear with respect to initial condition and input
- 2X input ⇒ 2X output when \( x(0) = 0 \)
Input/Output Performance

Return to system with inputs
- How does system response to changes in input values?

Transient response:
- What happens right after a new input is applied

Steady state response:
- What happens a long time after the input is applied

Stability vs input/output performance
- Systems that are close to instability typically exhibit poor input/output performance
- Nearly unstable systems (slow convergence) often exhibit “ringing” (highly oscillatory response to [non-periodic] inputs)
Step Response

Output characteristics in response to a “step” input

- Rise time: time required to move from 5% to 95% of final value
- Overshoot: ratio between amplitude of first peak and steady state value
- Settling time: time required to remain w/in p% (usually 2%) of final value
- Steady state value: final value at $t = \infty$
Important class of systems in many applications areas

\[ \ddot{q} + 2\zeta\omega_0 \dot{q} + \omega_0^2 q = u \quad \leftrightarrow \quad \frac{d}{dt} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\omega_0^2 & -2\zeta\omega_0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \]

- Analytical formulas exist for overshoot, rise time, settling time, etc
Frequency Response

Measure the *steady state* response of the system to sinusoidal input

- Example: audio amplifier – would like consistent ("flat") amplification between 20 Hz & 20,000 Hz
- Individual sinusoids are good *test signals* for measuring performance in many systems (eg, seasonal cycles in temperature)

Approach: plot input and output, measure *relative* amplitude and phase

- Generate response of system to sinusoidal output
- Gain = $\frac{A_y}{A_u}$
- Phase = $2\pi \cdot \frac{\Delta T}{T}$

May not work for *nonlinear* systems

- System nonlinearities can cause *harmonics* to appear in the output
- Amplitude and phase may not be well-defined
- For *linear* systems, frequency response is always well defined
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\[ u(t) \]
\[ q_1 \]
\[ m_1 \]
\[ k_1 \]
\[ q_2 \]
\[ m_2 \]
\[ k_2 \]
\[ k_3 \]
\[ c \]
Computing Frequency Responses

Technique #1: plot input and output, measure relative amplitude and phase

- Use `input_output_response` to generate response of system to sinusoidal output
- Gain = $A_y/A_u$
- Phase = $2\pi \cdot \Delta T/T$
- Note: In general, gain and phase will depend on the input amplitude

Technique #2 (linear systems): use Python `bode` command

- Assumes linear dynamics in state space form:
  \[ \dot{x} = Ax + Bu \]
  \[ y =Cx + Du \]
- Gain plotted on log-log scale
  - Traditional: $dB = 20 \log_{10} (gain)$
- Phase plotted on linear-log scale
Linearization Around an Equilibrium Point

\[ \dot{x} = f(x, u) \quad \dot{z} = Az + Bv \]
\[ y = h(x, u) \quad w = Cz + Dv \]

"Linearize" around \( x = x_e \)

\[
f(x_e, u_e) = 0 \quad y_e = h(x_e, u_e)
\]
\[
z = x - x_e \quad v = u - u_e \quad w = y - y_e
\]

\[
A = \left. \frac{\partial f}{\partial x} \right|_{(x_e, u_e)} \quad B = \left. \frac{\partial f}{\partial u} \right|_{(x_e, u_e)}
\]
\[
C = \left. \frac{\partial h}{\partial x} \right|_{(x_e, u_e)} \quad D = \left. \frac{\partial h}{\partial u} \right|_{(x_e, u_e)}
\]

Remarks

- In examples, this is often equivalent to small angle approximations, etc
- Only works near to equilibrium point

Full nonlinear model

Linear model (honest!)
Summary: Linear Systems

\[ \dot{x} = Ax + Bu \]
\[ y = Cx + Du \]
\[ x(0) = 0 \]

Properties of linear systems

- Linearity with respect to initial condition and inputs
- Stability characterized by eigenvalues
- Many applications and tools available
- Provide local description for nonlinear systems

\[ y(t) = Ce^{At}x(0) + \int_{0}^{t} Ce^{A(t-\tau)} Bu(\tau) d\tau + Du(t) \]