$\boxed{\text{CDS 110, Lecture 2.2: Dynamic Behavior}}$

RMM 8 Apr 2024

I. Questions from Monday's lecture

II. Review of nonlinear dynamics

process parameters

controller gains/params

A. $\dot{x} = F(x, \theta) \quad x \in \mathbb{R}^n, \ \theta \in \mathbb{R}^q \quad \left( \dot{x} = f(x, u, \alpha) + u = k(x, \beta) \rightarrow \dot{x} = F(x, \theta) \right)$

B. Eq pt: $F(x_{eq}, \theta) = 0 \Rightarrow$ start there, stay there

C. Stability: stable isL, asy stable, exp stable, unstable

D. Linear(ized) systems: $\dot{x} = A(\theta)x$, eigenvalues/eigenvectors

E. (optional) Lyapunov functions

III. Example: Internet congestion control ← FBS2e, Section 4.4

A. $\dot{x}_i = \underbrace{-b\frac{x_i^2}{2}}_{\text{backoff (drops)}} + \underbrace{(b_{max} - b)}_{\text{increase (RTT)}} \qquad \dot{b} = \left(\sum_{i=1}^{N} x_i\right) - c$

$c$ ↑ outgoing link

$x_i$ = source transmission rates

$b$ = router buffer size

B. Eq pts: $x_i = c/N \quad b = (\cdots)$

single source case: stable!

C. Linearization: $z = (b, x_1, \ldots)$

$$\dot{z} = \begin{bmatrix} 0 & 1 \ \cdots & 1 \\ -\left(\frac{x_{1,e}^2}{2}+1\right) & -bx_{1,e} & 0 \\ -\left(\frac{x_{2,e}^2}{2}+1\right) & 0 & -bx_{2,e} \end{bmatrix} z = \begin{bmatrix} 0 & 1 & 1 \\ -\left(\frac{c^2}{2N}+1\right) & -\frac{bc}{N} & 0 \\ -\left(\frac{c^2}{2N}+1\right) & 0 & -\frac{bc}{N} \end{bmatrix}$$

IV (optional) Limit cycles

IV Discrete time systems

A. $x[k+1] = F(x[k], \theta)$ - definitions of stability unchanged

B. $x[k+1] = A(\theta)\, x[k]$ – $|\lambda_i| < 1, \ldots$

C. Example: predator-prey

food supply

lynx eats hares

$$H[k+1] = H[k] + b_h(u)\, H[k] - a\, L[k]\, H[k]$$

$$L[k+1] = L[k] + c\, L[k]\, H[k] - d_\ell\, L[k] \leftarrow \text{lynx die}$$

lynx eats hares

Eq pts: $(b_h(u) - a L_e)\, H_e = 0 \Rightarrow L_e = \frac{b_h(u_e)}{a}$

$c L_e H_e - d_\ell L_e = 0 \Rightarrow H_e = \frac{d_\ell}{c}$

Stability: $x[k+1] = A\,x[k]$

$(u = 0)$

$$A = \begin{bmatrix} 1 + b_h - aL_e & -aH_e \\ cL_e & 1 + cH_e - d_\ell \end{bmatrix} = \begin{bmatrix} 1 & -aH_e \\ cL_e & 1 \end{bmatrix}$$

Eigenvalues: $(s-1)^2 + acL_eH_e = s^2 - 2s + (1 + acL_eH_e)$

$\Rightarrow \lambda = 1 \pm \sqrt{-acL_eH_e} \Rightarrow |\lambda_i| > 1 \Rightarrow$ unstable