CDS 110/ChE 105: Lecture 2.1

Modeling and Stability Analysis of Dynamical Systems

Manisha Kapasiawala

8 April 2024

Learning objectives

By the end of today's lecture, you will be able to:

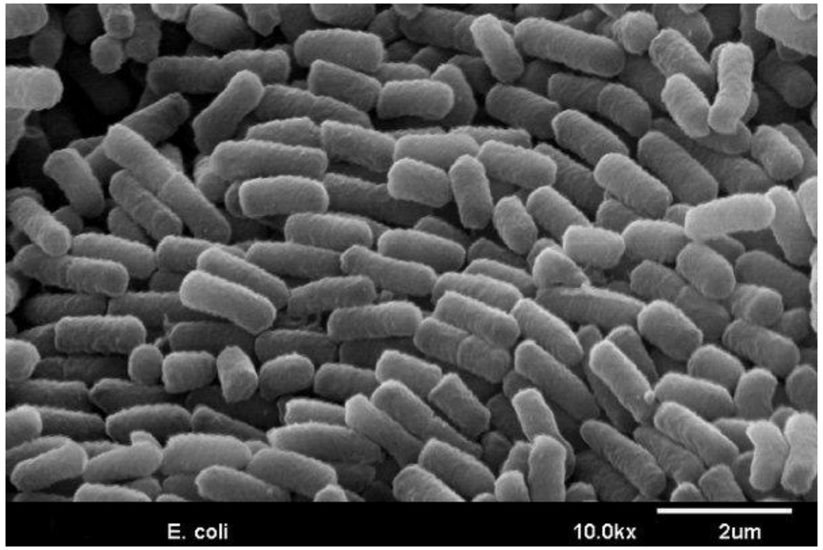
- understand considerations in **choosing an appropriate model** to describe a dynamical system
- **interpret a state space model** (for continuous- and discrete-time systems)
- **interpret phase portraits** to identify equilibrium points and limit cycles
- **perform stability analysis** and distinguish between different types of stability

Models can help us understand how a dynamical system will behave

The choice of model depends on:

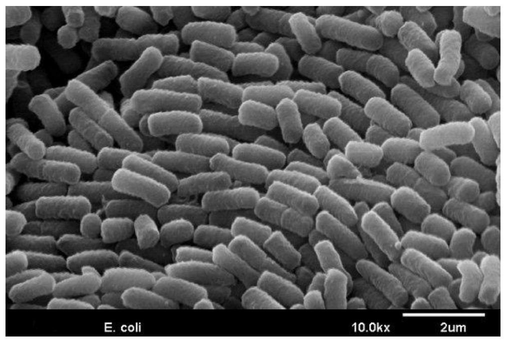
- the question we are trying to answer about the system
- the context of the model (its assumptions, in which regime it will be used, what we can measure, etc.)

Models don't have to be perfect to be informative, particular in the context of feedback control.



E. coli
(K. Keerthana, 2016)

One system can be modeled in many ways

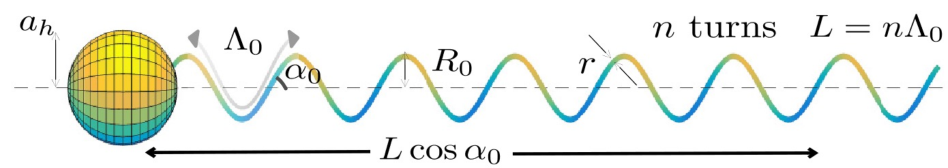


E. coli
(K. Keerthana, 2016)

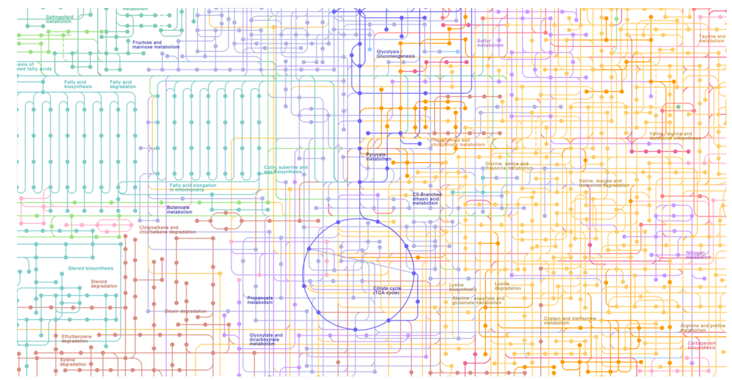
"How fast can this microbe swim in a particular environment?"
fluid dynamics

"How fast does this microbe consume glucose?"
chemical reaction network

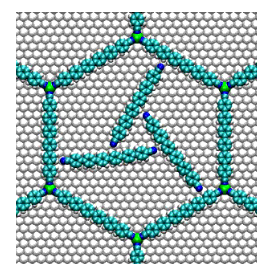
"How do this microbe's proteins interact?"
molecular dynamics



https://doi.org/10.1103/PhysRevE.99.053107



https://www.genome.jp/pathway/map01100



https://commons.wikimedia.org/wiki/File:MD_rotor_250K_1ns.gif

In control, we use state space models to define systems

$$\frac{dx}{dt} = f(x, u)$$

$$y = h(x, u)$$

State vector: $x \in \mathbb{R}^n$

Inputs: $u \in \mathbb{R}^p$

Measured outputs : $y \in \mathbb{R}^q$

$f : \mathbb{R}^n \times \mathbb{R}^p \rightarrow \mathbb{R}^n$ and $h : \mathbb{R}^n \times \mathbb{R}^p \rightarrow \mathbb{R}^q$ are smooth mappings

If f and h are linear in x and u, then the model takes on a linear form

$$\frac{dx}{dt} = Ax + Bu$$

$$y = Cx + Du$$

Dynamics matrix: $A \in \mathbb{R}^{n\text{x}n}$

Control matrix: $B \in \mathbb{R}^{n\text{x}p}$

Sensor matrix: $C \in \mathbb{R}^{\text{qx}n}$

Direct term: $D \in \mathbb{R}^{q\text{x}p}$

If f and h do not explicitly depend on time, this model is a linear time-invariant (LTI) system.

Example: spring-mass system with damping

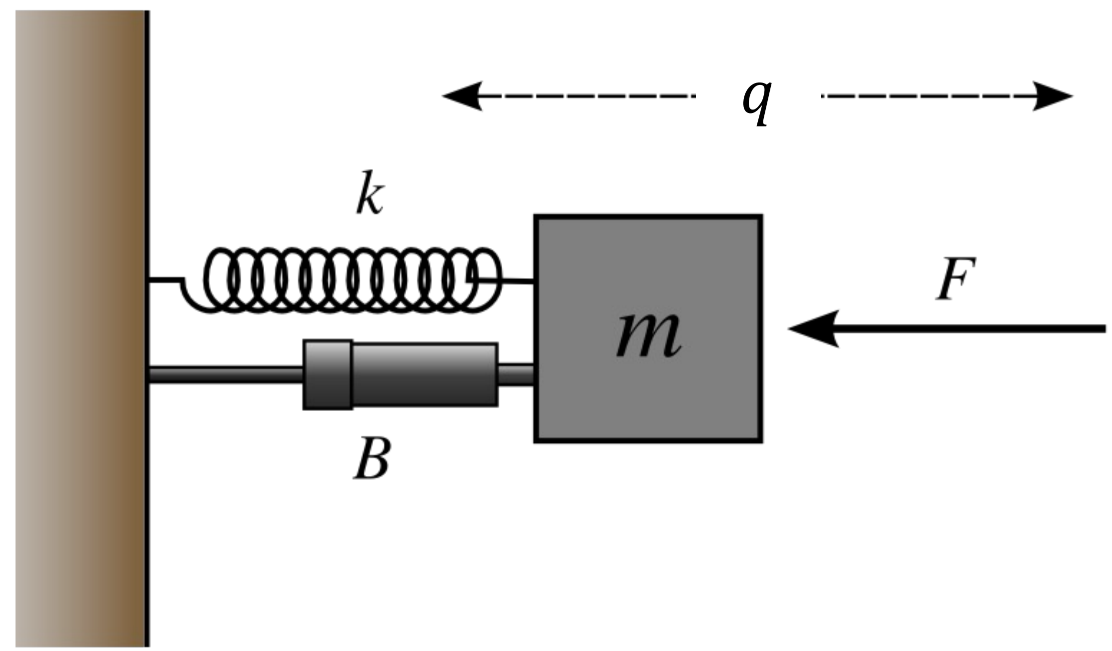
$$m\ddot{q} + c(\dot{q}) + kq = u$$

↓ Define "state", output

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \dot{q} \\ q \end{bmatrix}, \qquad y = q$$

↓ Convert to state space → system of first-order ODEs

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -c/m & -k/m \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1/m \\ 0 \end{bmatrix}u, \qquad y = \begin{bmatrix} 0 & 1 \end{bmatrix}x$$



A model can be converted into state space form even if the original equations are not first-order ODEs

$$\frac{d^n y}{dt^n} + a_1 \frac{d^{n-1} y}{dt^{n-1}} + a_2 \frac{d^{n-2} y}{dt^{n-2}} + \cdots + a_n y = u$$

Define state

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} = \begin{bmatrix} d^{n-1}y/dt^{n-1} \\ d^{n-2}y/dt^{n-2} \\ \vdots \\ dy/dt \\ y \end{bmatrix}$$

Convert to state space

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} = \begin{bmatrix} -a_1 x_1 - \cdots - a_n x_n \\ x_1 \\ \vdots \\ x_{n-2} \\ x_{n-1} \end{bmatrix} + \begin{bmatrix} u \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix}$$

$$y = x_n$$

We can also represent discrete-time systems as state space models

General form

$$x[k+1] = f(x[k], u[k])$$

$$y[k] = h(x[k], u[k])$$

⟶

Linear form

$$x[k+1] = Ax[k] + Bu[k]$$

$$y[k] = Cx[k] + Dx[k]$$

State vector (at time k): $x[k] \in \mathbb{R}^n$,
Inputs: $u[k] \in \mathbb{R}^p$,
Measured outputs : $y[k] \in \mathbb{R}^q$

Example: predator-prey system

Number of hares: H
Number of lynxes: L

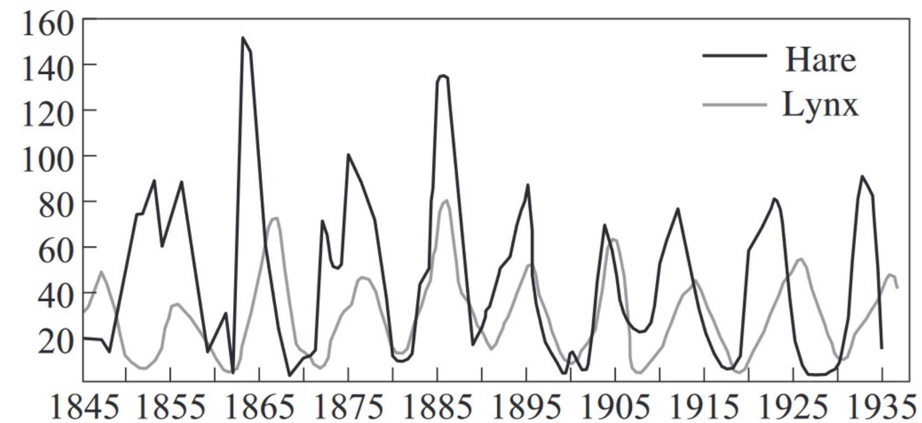


Figure 3.7, FBS2e

$$H_{\text{next time point}} = H_{\text{current time point}} + H_{\text{born}} f_{\text{hare food}}(u) - H_{\text{eaten}} f_{\text{predation}}(L)$$

$$L_{\text{next time point}} = L_{\text{current time point}} + L_{\text{born}} f_{\text{lynx food}}(H) - f_{\text{mortality}}(L)$$

Example: predator-prey system

Number of hares: H
Number of lynxes: L

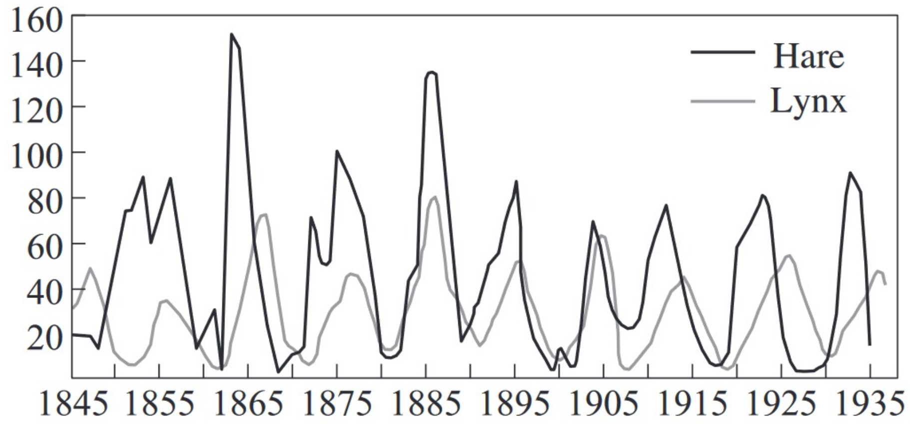


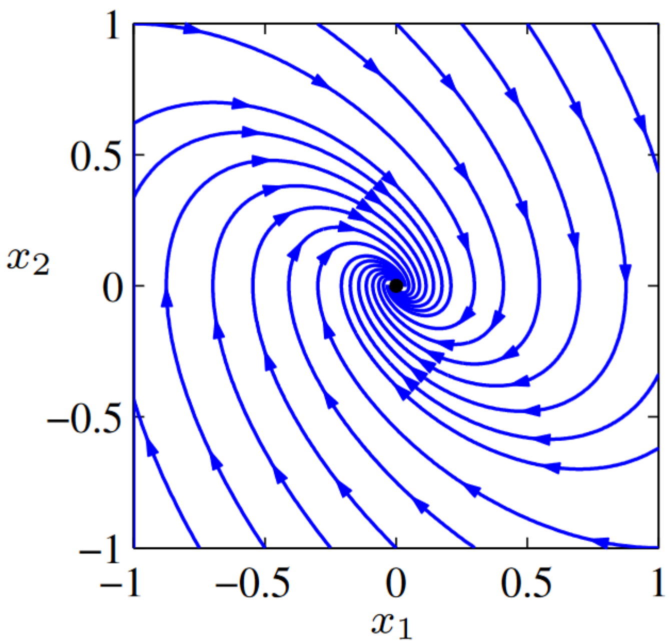
Figure 3.7, FBS2e

$$H[k+1] = H[k] + b_h(u)H[k] - aL[k]H[h]$$

$$L[k+1] = L[k] + cL[k]H[k] - d_l L[k]$$

Once we have a state space model, we can use it to perform stability analysis

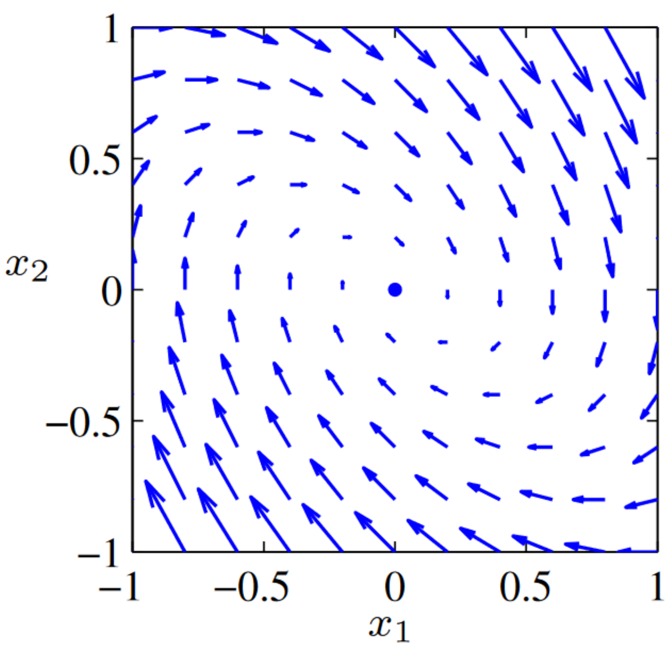
Qualitative



Quantitative

$$\lambda(A) := \{s \in \mathbb{C} : \det(sI - A) = 0\}$$

Systems with two state variables ($x \in \mathbb{R}^2$) can be analyzed via phase portraits



(a) Vector field

Figure 5.3, FBS2e

Phase portraits can show equilibrium points, which represent stationary conditions for dynamics

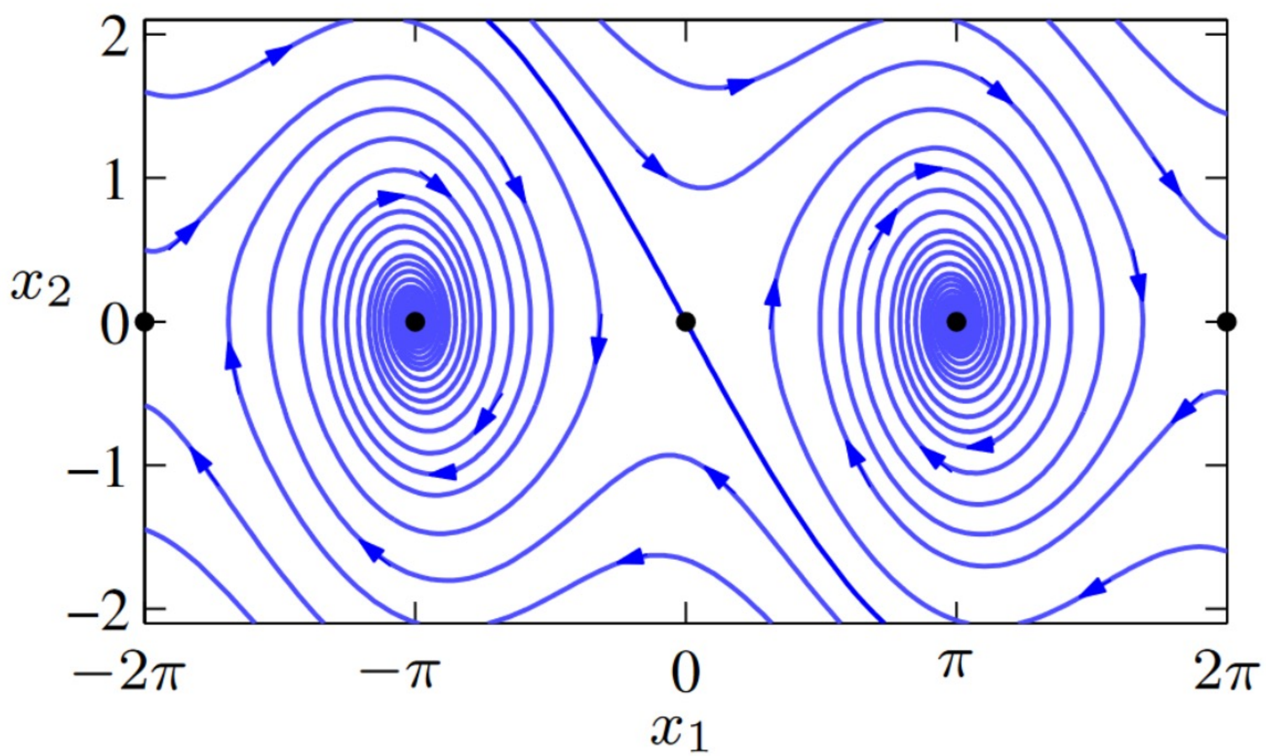


Figure 5.3, FBS2e

For a dynamical system

$$\frac{dx}{dt} = F(x)$$

equilibrium points occur at $x = x_e$ where

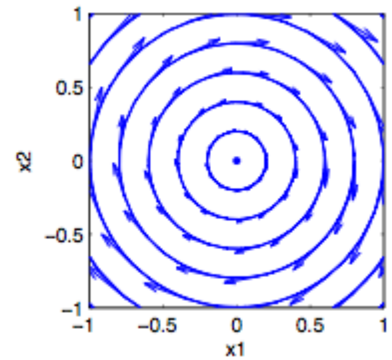
$$F(x_e) = 0$$

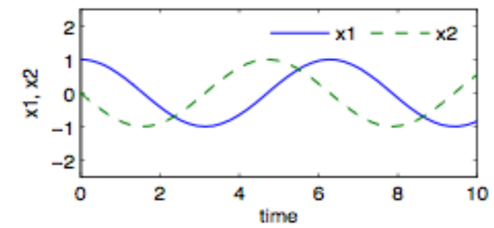
Stability of Equilibrium Points

An equilibrium point is:

Stable if initial conditions that start near the equilibrium point, stay near

- Also called "stable in the sense of Lyapunov"

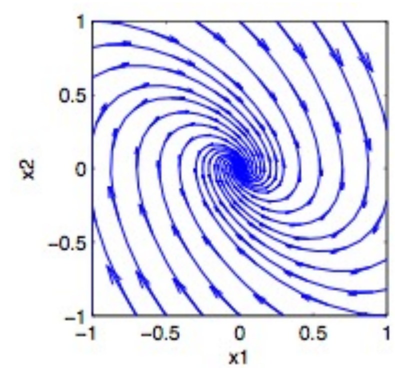


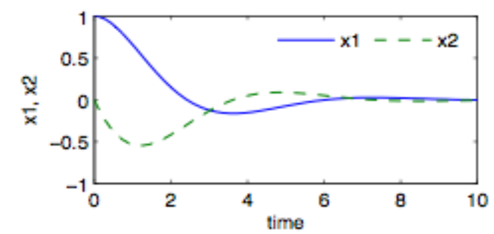


$$\|x(0) - x_e\| < \delta \implies \|x(t) - x_e\| < \epsilon$$

Asymptotically stable if all nearby initial conditions converge to the equilibrium point

- Stable + converging

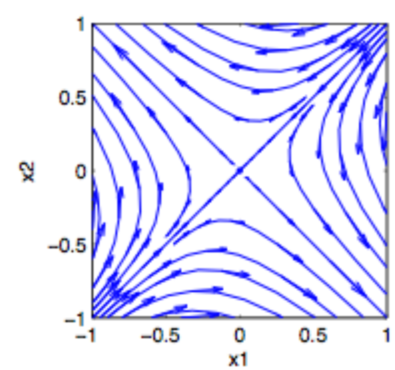


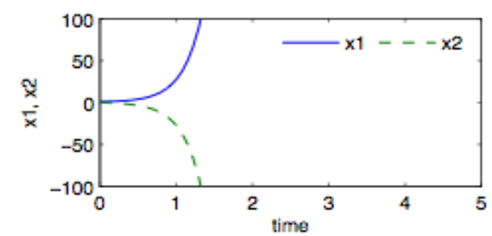


$$\lim_{t \to \infty} x(t) = x_e \quad \forall \|x(0) - x_e\| < \epsilon$$

Unstable if some initial conditions diverge from the equilibrium point

- May still be some initial conditions that converge





Systems with oscillatory behavior may exhibit a stable limit cycle, a stable *trajectory* in x_1-x_2 space

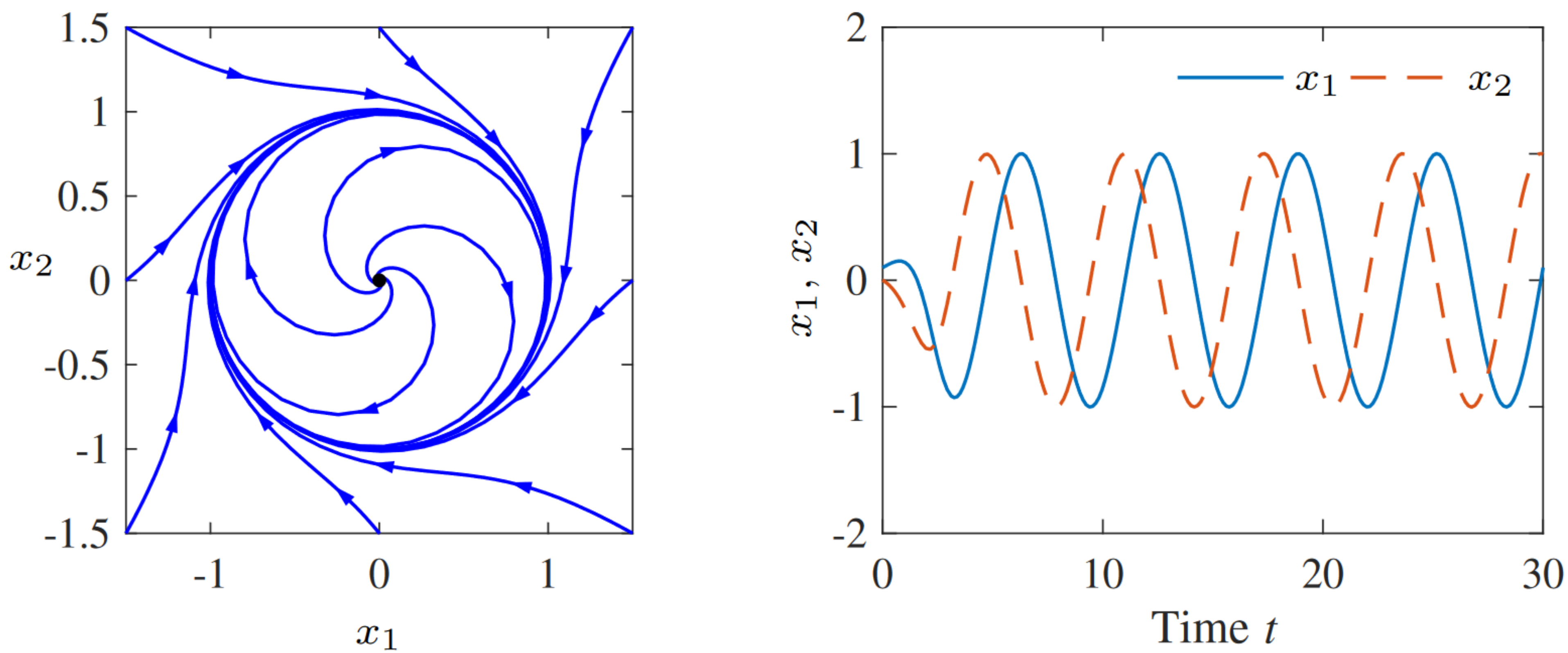


Figure 5.5, FBS2e

For linear systems, we can compute eigenvalues to determine stability analytically

Linear dynamical system with state : $x \in \mathbb{R}^n$

$$\frac{dx}{dt} = Ax \qquad x(0) = x_0,$$

Stability determined by the eigenvalues $\lambda(A) = \{s \in \mathbb{C} : \det(sI - A) = 0\}$

- Simplest case: diagonal *A* matrix (all eigenvalues are real)

$$\frac{dx}{dt} = \begin{bmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{bmatrix} x$$

$$\dot{x}_i = \lambda_i x_i$$

$$x_i(t) = e^{\lambda_i t} x(0)$$

  - System is asy stable if $\lambda_i < 0$

- Block diagonal case (complex eigenvalues)

$$\frac{dx}{dt} = \begin{bmatrix} \sigma_1 & \omega_1 & & 0 & 0 \\ -\omega_1 & \sigma_1 & & 0 & 0 \\ 0 & 0 & \ddots & \vdots & \vdots \\ 0 & 0 & & \sigma_m & \omega_m \\ 0 & 0 & & -\omega_m & \sigma_m \end{bmatrix} x$$

$$x_{2j-1}(t) = e^{\sigma_j t}\big(x_i(0)\cos\omega_j t + x_{i+1}(0)\sin\omega_j t\big)$$
$$x_{2j}(t) = e^{\sigma_j t}\big(x_i(0)\sin\omega_j t - x_{i+1}(0)\cos\omega_j t\big)$$

  - System is asy stable if $\operatorname{Re}\lambda_i = \sigma_i < 0$

For nonlinear systems, we can first make a linear approximation of the system

Asymptotic stability of the linearization implies *local* asymptotic stability of equilibrium point

- Linearization around equilibrium point captures "tangent" dynamics

$$\dot{x} = \underbrace{F(x_e)}_{0} + \left.\frac{\partial F}{\partial x}\right|_{x_e} (x - x_e) + \text{higher order terms} \quad \xrightarrow{approx} \quad \begin{aligned} z &= x - x_e \\ \dot{z} &= Az \end{aligned}$$

- If linearization is *unstable,* can conclude that nonlinear system is locally unstable
- If linearization is *stable* but not *asymptotically stable,* can't conclude anything about nonlinear system:

$$\dot{x} = \pm x^3 \quad \xrightarrow{linearize} \quad \dot{x} = 0$$

- linearization is stable (but not asy stable)
- nonlinear system can be asy stable or unstable

Local approximation particularly appropriate for control systems design

- Control often used to *ensure* system stays near desired equilibrium point
- If dynamics are well-approximated by linearization near equilibrium point, can use this to design the controller that keeps you there (!)

Local versus Global Behavior

Stability is a *local* concept

- Equilibrium points define the local behavior of the dynamical system
- Single dynamical system can have stable *and* unstable equilibrium points

Region of attraction

- Set of initial conditions that converge to a given equilibrium point