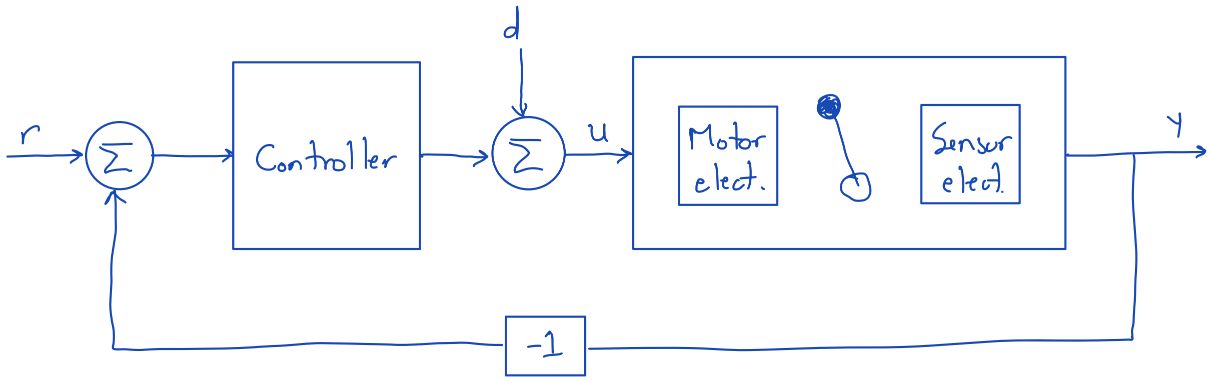
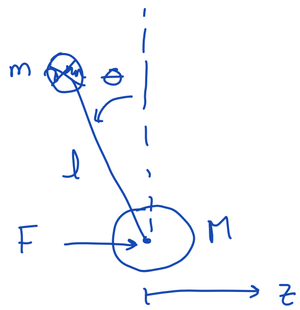
CDS 110, Lecture 1.2: Feedback Principles



Outline

I. Problem setup – MinSeg + block diagram
II. Modeling and stability ⎫
III. Proportional control ⎭ stability
IV. Derivative control] performance
V. Disturbance rejection – feedforward + integral
VI. Frequency response

MinSeg dynamics



$$m l^2 \ddot\theta = -\overset{\text{damping (small)}}{b}\,\dot\theta + m g l \sin\theta + l F \cos\theta$$

$$M \ddot z = \cdots \ (?)$$

Focus on θ dynamics:

$$\ddot\theta + \frac{b}{m l^2}\dot\theta - \frac{g}{l}\sin\theta = \frac{1}{m l} F\cos\theta \rightsquigarrow \ddot y + \alpha \dot y + \beta y = \gamma F$$

$\alpha > 0 \quad \beta < 0$

Rewrite in "first order form"

$$x = \begin{bmatrix} \theta \\ \dot\theta \end{bmatrix} \qquad \frac{dx}{dt} = \begin{bmatrix} x_2 \\ -\frac{b}{m l^2} x_2 + \frac{g}{l}\sin x_1 + \frac{1}{m l} F \end{bmatrix} \approx \underbrace{\begin{bmatrix} 0 & 1 \\ \frac{g}{l} & -\frac{b}{m l^2} \end{bmatrix}}_{A = \begin{bmatrix} 0 & 1 \\ -\beta & -\alpha \end{bmatrix}} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \\ \frac{1}{m l} \end{bmatrix}}_{B = \begin{bmatrix} 0 \\ \gamma \end{bmatrix}} u$$

To study stability, look at characteristic eq

$$s^2 + \alpha s + \beta = 0 \Rightarrow \lambda = \frac{-\alpha \pm \sqrt{\alpha^2 - 4\beta}}{2} \approx -\frac{\alpha}{2} \pm \beta \quad (\text{assuming } \alpha^2 \ll 4\beta)$$

$\alpha > 0 \quad \beta < 0$

Solution to homogeneous (linearized) eq $(F = 0)$:

$$y = \underset{\text{stable}}{a_1 e^{(-\frac{\alpha}{2} + \beta)t}} + \underset{\text{unstable}}{a_2 e^{(-\frac{\alpha}{2} - \beta)t}} \qquad (\alpha > 0 \text{ but small, } \beta < 0)$$

Stabilization with feedback

if $\beta' > \frac{\alpha^2}{4} \Rightarrow$ imaginary

Suppose we set $F = -k_p\theta$ (push in direction of "lean")

$$\ddot{y} + \alpha\dot{y} + \beta y = -\gamma k_p y \Rightarrow \ddot{y} + \alpha\dot{y} + \underbrace{(\beta + \gamma k_p)}_{\beta'} y = 0 \Rightarrow \lambda = \frac{-\alpha \pm \sqrt{\alpha^2 - 4\beta'}}{2}$$

System is now stable   choose k_p s.t. $\rightarrow \beta' > 0$ $\approx -\frac{\alpha}{2} \pm i\sqrt{4\beta' - \alpha^2}$

$$y = a_1 e^{-\alpha t/2} \sin\omega t + a_2 e^{-\alpha t/2} \cos\omega t \qquad \omega = \sqrt{4\beta' - \alpha^2}$$

stable   stable

Note that if α is small, convergence might be slow...

more generally

$\dot{x} = Ax + Bu \quad x = \begin{bmatrix}\theta \\ \dot{\theta}\end{bmatrix}$

$u = -Kx$

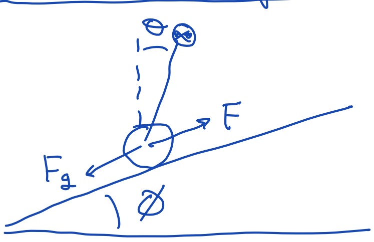
More complicated: $F = -k_p\theta - k_d\dot{\theta}$ (PD)

$$\ddot{y} + \underbrace{(\alpha + \gamma k_d)}_{\text{set w/ } k_d}\dot{y} + \underbrace{(\beta + \gamma k_p)}_{\text{set w/ } k_p} y = 0 \Rightarrow$$

We can "design" dynamics (eigenvalue placement)

$$y = a_1 e^{-\lambda t}\cos\omega t + a_2 e^{-\lambda t}\sin\omega t$$

Disturbance rejection



Roughly speaking   "disturbance"

$$ml\ddot{\theta} + b\dot{\theta} - mgl\sin\theta = l(F - mg\sin\phi)\cos\theta$$

(some work)

$$\ddot{y} + \alpha\dot{y} + \beta y = \gamma F + \underbrace{\left(-\frac{g}{l^2}\sin\phi\right)}_{d}$$

What is the effect of d?

$$\ddot{y} + \alpha\dot{y} + \beta' y = d \qquad \text{(remember } y = \theta\text{)}$$

Case 1: transient (or periodic d)

- Feedback attenuates disturbances (potentially skip to freq response)

Case 2: steady state disturbance

For $d \neq 0$, if $\dot{y} = \ddot{y} = 0 \Rightarrow y = \frac{d}{\beta'} \Rightarrow F = \frac{k_p}{\beta'} d$

If $\frac{k_p}{\beta'}\left(-\frac{g}{l^2}\sin\phi\right) - mg\sin\phi \neq 0 \Rightarrow$ continuously accelerating

Experiment: see what happens (doesn't match prediction)

Integral Feedback

Q: how do we fix this?

A1: Feed Forward $\quad F = \underbrace{\frac{mg\sin\phi}{l}}_{\substack{\text{force required to balance gravity} \\ \text{("feedforward")}}} \underbrace{- k_p\theta}_{\text{feedback}}$

Now dynamics are

$$\ddot{y} + \alpha\dot{y} + \beta y = \underbrace{\frac{g}{l^2}\sin\phi - \frac{g}{l^2}\sin\phi}_{=0!} - k_p\theta$$

Problem: need to know m, g & l exactly $\Rightarrow$ not robust!

A2: Integral feedback

$$F = -k_p\theta - k_i\int_0^t (\theta(\tau) - \theta_d)\,d\tau$$

Rough intuition: integrator will "compute" required bias. Note that *if* we stabilize, then $\theta \to 0$. But integral of θ can be nonzero.

Turns out that integral will converge to $\frac{mg\sin\phi}{l k_i}$ *without* prior knowledge of m, g, l (or even k_i!)

This property makes integral feedback one of *the most common* types of feedback

PID = proportional-integral-derivative

Additional topics, if time

Transfer functions

Assume *linear* system

$$u \longrightarrow \boxed{P} \longrightarrow y$$

$$\underbrace{u = A \sin \omega t}_{\text{sinusoid}} \rightarrow y(t) = \text{transient} + \underbrace{B \sin(\omega t + \phi)}_{\text{sinusoid}}$$

transient: goes to zero if system is stable

More generally,

$$\underbrace{u = A e^{st}}_{\text{complex \#}} \rightarrow y = \cancel{\text{transient}}^{\,0} + \underbrace{B(s) e^{st}}_{\text{complex \#}}$$

$$e^{st} = e^{(\sigma + j\omega)t} = e^{\sigma t}(\cos \omega t + i \sin \omega t)$$

Transfer function $\quad u = e^{st} \rightarrow y = H(s)\, e^{st}$

$H(s)$: $B(s)/A$

Plan for Friday: work through this example in Python

Announcements

1. Everyone should be signed up for Piazza & Gradescope. Email Richard if not.
2. Reminder: if you want to be course ombuds, send email today
3. For Fri: if you have a laptop or iPad, feel free to bring it (web-based)