CALIFORNIA INSTITUTE OF TECHNOLOGY
Control and Dynamical Systems

CDS 101

R. M. Murray — **Problem Set #4** — Issued: 19 Oct 2015
Fall 2015 — Due: 2 pm, 28 Oct 2015

Note: In the upper left hand corner of the *second* page of your homework set, please put the number of hours that you spent on this homework set (including reading).

Reminder: You get two grace periods of no more than 2 days each for late homework. After that, late homework will not be accepted without a note from the dean or the health center.

1. Consider the normalized dynamics of an inverted pendulum described by the model

$$\frac{dx_1}{dt} = x_2, \qquad \frac{dx_2}{dt} = \sin x_1 + u \cos x_1,$$

where x_1 is the angular deviation from the upright position (θ), x_2 is the angular rate ($\dot{\theta}$) and u is the (scaled) acceleration of the pivot, as shown in Figure 5.16a.

(a) Show that the linearization of the dynamics around the (upward-pointing) equilibrium point $x_e = (0, 0)$ is given by

$$\frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u = Ax + Bu.$$

(b) Compute the eigenvalues of the linearized dynamics and show that this equilibrium point is unstable.

(c) Compute the reachability matrix for the system and show that the linearized system is reachable.

(d) Determine a state feedback control law of the form $u = -Kx$ that gives a closed loop system with with the characteristic polynomial $s^2 + 2\zeta_0\omega_0 s + \omega_0^2$.

(e) Set $\omega_0 = 1$ and $\zeta_0 = 0.5$. Compute the eigenvalues for the resulting closed loop system and verify that the equilibrium point is no longer unstable.

(f) Simulate the response of the original nonlinear system from a set of initial conditions that each correspond to the system starting at rest ($\dot{\theta} = 0$) with initial angle $\theta(0)$ equal to 0.1 rad, 0.5 rad, 1 rad and 2 rad.

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1. Consider the normalized, linearized inverted pendulum which is described by

$$\frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u = Ax + Bu, \qquad y = \begin{pmatrix} 1 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = Cx$$

Determine a state feedback and reference gain $u = -Kx + k_r r$ that gives a closed loop system with unit static gain (steady-state output $y = r$) and with the characteristic polynomial $s^2 + 2\zeta_0\omega_0 s + \omega_0^2$.

2. Consider the system

$$\frac{dx}{dt} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} x + \begin{pmatrix} 1 \\ 0 \end{pmatrix} u, \qquad y = \begin{pmatrix} 1 & 0 \end{pmatrix} x,$$

with the control law

$$u = -k_1 x_1 - k_2 x_2 + k_r r.$$

Compute the rank of the reachability matrix for the system and show that eigenvalues of the system cannot be assigned to arbitrary values.

Note: You sould *explicitly* show that not all eigenvalues can be assigned arbitrarily; don't just cite Theorem 6.1.)

3. (Åström and Murray, Exercise 7.10) Let $A \in \mathbb{R}^{n\times n}$ be a matrix with characteristic polynomial $\lambda(s) = \det(sI - A) = s^n + a_1 s^{n-1} + \cdots + a_{n-1}s + a_n$. Assume that the matrix A can be diagonalized and show that it satisfies

$$\lambda(A) = A^n + a_1 A^{n-1} + \cdots + a_{n-1}A + a_n I = 0,$$

Use the result to show that A^k, $k \geq n$, can be rewritten in terms of powers of A of order less than n.

Note: Assume that the A matrix is diagonalizable (the theorem is valid but hard to prove with a non-trivial Jordan form).

4. (Åström and Murray, Exercise 7.13) Consider the Whipple bicycle model given by equation (4.7) in Section 4.2. Using the parameters from the companion web site, the model

is unstable at the velocity $v = 5$ m/s and the open loop eigenvalues are -1.84, -14.29 and $1.30 \pm 4.60i$. Find the gains of a controller that stabilizes the bicycle and gives closed loop eigenvalues at -2, -10 and $-1 \pm i$. Simulate the response of the system to a step change in the steering reference of 0.002 rad.

Next, find the controller gains corresponding to choosing the final pair of complex poles at $-2 \pm 2i$ and $-5 \pm 5i$. In addition to calculating the state feedback gains, make sure to solve for the referene gain k_r as well. For each case, simulate the response to a step change in the steering reference of 0.002 rad and plot both the steering angle and the torque command.

Note: Download the file `bike_linmod.m` from the course web site, which contains the parameters for the bicycle and generates the matrices M, C, K_0 and K_2 in equation (4.7) of the text.

Find the controller gains corresponding to choosing the final pair of complex poles at $-1 \pm i$ as stated in the text, and also with these poles at $-2 \pm 2i$ and $-5 \pm 5i$. In addition to calculating the state feedback gains, solve for the reference gain k_r as well! When simulating the response to a step change in the steering reference of 0.002 rad, plot both the steering angle output δ and the torque command.