CALIFORNIA INSTITUTE OF TECHNOLOGY
Control and Dynamical Systems

CDS 101

Problem Set #3

D. G. MacMynowski
Fall 2008

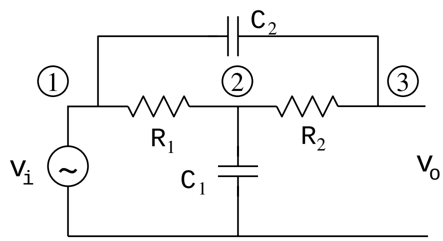
Issued: 13 Oct 08
Due: 20 Oct 08

Note: In the upper left hand corner of the *second* page of your homework set, please put the number of hours that you spent on this homework set (including reading).

1. For each of the following linear systems, determine whether the origin is asymptotically stable and, if so, plot the step response and frequency response for the system. If there are multiple inputs or outputs, plot the response for each pair of inputs and outputs.

   (a) *Coupled mass spring system.* Consider the coupled mass spring system from Example 5.5 with $m = 250$, $k = 50$ and $c = 10$. The input $u(t)$ is the force applied to the right-most spring.

   (b) *Bridged Tee Circuit.* Consider the following electrical circuit, with input v_i and output $y = v_o$.



   The dynamics are given by

$$\frac{d}{dt}\begin{pmatrix} v_{c1} \\ v_{c2} \end{pmatrix} = \begin{pmatrix} -\frac{1}{C_1}\left(\frac{1}{R_1}+\frac{1}{R_2}\right) & -\frac{1}{C_1 R_2} \\ -\frac{1}{C_2 R_2} & -\frac{1}{C_2 R_2} \end{pmatrix}\begin{pmatrix} v_{c1} \\ v_{c2} \end{pmatrix} + \begin{pmatrix} \frac{1}{C_1}\left(\frac{1}{R_1}+\frac{1}{R_2}\right) \\ \frac{1}{C_2 R_2} \end{pmatrix} v_i,$$

$$y = \begin{pmatrix} 0 & 1 \end{pmatrix}\begin{pmatrix} v_{c1} \\ v_{c2} \end{pmatrix} + v_i,$$

   where v_{c1} and v_{c2} are the voltages across the two capacitors. Assume that $R_1 = 100\ \Omega$, $R_2 = 100\ \Omega$ and $C_1 = C_2 = 1 \times 10^{-6}$ F.

2. Consider the balance system described in Example 2.1 of the text, using the following parameters:

$$M = 10 \text{ kg}, \quad m = 80 \text{ kg}, \quad J = 100 \text{ kg m}^2, \quad c = 0.1 \text{ N/m/sec}, \quad l = 1 \text{ m}, \quad \gamma = 0.01 \text{ Nms}, \quad g = 9.8 \text{ m/s}^2.$$

   This system has been modeled in SIMULINK in the file `balance_simple.mdl`, available from the course web page. (Note: in the SIMULINK model, the output has been set to include all of the states ($y = x$). You will need this for part (c) below.)

(a) Use the MATLAB `linmod` command to numerically compute the linearization of the original nonlinear system at the equilibrium point $(x, \theta, \dot{x}, \dot{\theta}) = (0, 0, 0, 0)$. Compare the eigenvalues of the analytical linearization (from the text) to those of the one you obtained with `linmod` and verify they agree. (Make sure to look at the errata sheet for the text; there are some small glitches in the equations listed in both Example 2.1 and Example 6.7.)

(b) We can design a stabilizing control law for this system using "state feedback", which is a control law of the form $u = -Kx$ (we will learn about this more next week). The closed loop system under state feedback has the form

$$\frac{dz}{dt} = (A - BK)z.$$

Show that the following state feedback stabilizes the linearization of the inverted pendulum on a cart: $K = [-15.3\ 1730\ \ -50\ 443]$.

(c) Now build a simulation for the closed loop, *nonlinear* system in SIMULINK. Use the file `balance_simple.mdl` for the nonlinear equations of motion in it (you should look in the file and try to understand how it works). Simulate several different initial conditions and show that the controller *locally* asymptotically stabilizes the system to x_e from these initial conditions. Include plots of a representative simulation for an initial condition that is in the region of attraction of the controller and one that is outside the region of attraction.

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CDS 110a

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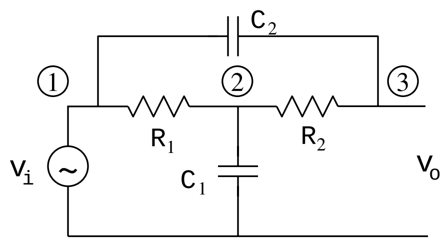

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3. Åström and Murray, Exercise 5.7

4. Åström and Murray, Exercise 5.8, parts (a) through (c). For part (a), note that the equation can also be written as.

$$x[k] = CA^k x_0 + \sum_{i=0}^{k-1} CA^i Bu[k-1-i] + Du[k]$$

For part (b), you can assume that the matrix A has a full basis of eigenvectors. For part (c), use input $u[k] = \sin \omega k$.

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CDS 210

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2. Åström and Murray, Exercise 5.6
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   For part (b), you can assume that the matrix A has a full basis of eigenvectors. For part (c), use input $u[k] = \sin \omega k$.

4. Using LMI to prove stability.

   (a) Download YALMIP from the web (see instructions on the class wiki.)

   Consider the following system

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_n \\ \dot{x}_{n+1} \end{bmatrix} = \begin{bmatrix} -c_1 & r_2 & \cdots & 0 & \sigma \\ k_1 & -c_2 & & 0 & 0 \\ \vdots & & \cdots & r_n & \vdots \\ 0 & 0 & & -c_n & r_{n+1} \\ 0 & 0 & \cdots & bk_n & -d_{n+1} - a\sigma \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \\ x_{n+1} \end{bmatrix}$$

   where $c_i = r_i + k_i, \quad b = 2, \quad a = 1, \quad d_{n+1} = 1 + r_{n+1}$.

   (b) Let $\sigma = -0.2$ and $n = 50$. Generate random values for the parameters $r_i, k_i > 0$. Using the Lyapunov inequality $AP + PA^T < -Id$ prove that the system is stable. Check that P is positive definite. What if we take $\sigma = -1$.

   (c) Let $r_i = 0.2, \quad k_i = 0.8, \quad \forall i$ and $n = 10$. Using the Lyapunov inequality find the smallest negative σ for which the system is still stable (Try different values of σ, search for P, and check that the problem is feasable.)