Feedback Systems: Notes on Linear Systems Theory

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These notes are a supplement for the second edition of *Feedback Systems* by Åström and Murray (referred to as FBS2e), focused on providing some additional mathematical background and theory for the study of linear systems.

Chapter 3

Reachability and Stabilization

Preliminary reading The material in this chapter extends the material in Chapter 7 in FBS2e. Readers should be familiar with the material in Sections 7.1 and 7.2 in preparation for the more advanced concepts discussed here.

3.1 Concepts and Definitions

Consider an input/output dynamical system $\mathcal{D} = (\mathcal{U}, \Sigma, \mathcal{Y}, s, r)$ as defined in Section 1.2.

Definition 3.1 (Reachability)**.** A state x_f is *reachable from* x_0 *in time* T if there exists an input $u : [0, T] \to \mathbb{R}^m$ such that $x_\text{f} = s(T, t_0, x_0, u)$.

If x_f is reachable from x_0 in time T we will write

$$x_0 \underset{T}{\rightsquigarrow} x_\text{f}$$

or sometimes just $x_0 \rightsquigarrow x_\text{f}$ if there exists some T for which x_f is reachable from x_0 in time T. The set of all states that are reachable from x_0 in time less than or equal to T is written as

$$\mathcal{R}_{\leq T}(x_0) = \{x_\text{f} \in \Sigma : x_0 \underset{\tau}{\rightsquigarrow} x_\text{f} \text{ for some } \tau \leq T\}.$$

Definition 3.2 (Reachable sytem)**.** An input/output dynamical system $\mathcal{D}$ is *reachable* if for every $x_0, x_\text{f} \in \Sigma$ there exists $T > 0$ such that $x_0 \underset{T}{\rightsquigarrow} x_\text{f}$.

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The notion of reachability captures the property that we can reach a any final point x_f starting from x_0 with some choice of input $u(\,\cdot\,)$. In many cases, it will be not be possible to reach *all* states x_f but it may be possible to reach an open neighborhood of such points.

Definition 3.3 (Small-time local controllability)**.** A system is *small-time locally controllable* (STLC) if for any $T > 0$ the set $\mathcal{R}_{\leq T}(x_0)$ contains a neighborhood of x_0.

The notions of reachability and (small-time local) controllability hold for arbitrary points in the state space, but we are often most interested in equilibrium points and our ability to stabilize a system via state feedback. To define this notion more precisely, we specialize to the case of a state space control systems whose dynamics can be written in the form

$$\frac{dx}{dt} = f(x, u), \qquad x(0) = x_0. \tag{S3.1}$$

Definition 3.4 (Stabilizability)**.** A control system with dynamics (S3.1) is *stabilizable* at an equilibrium x_e if there exists a control law $u = \alpha(x, x_e)$ such that

$$\frac{dx}{dt} = f(x, \alpha(x, x_e)) =: F(x)$$

is locally asymptotically stable at x_e.

The main distinction between reachability and stabilizability is that there may be regions of the state space that are not reachable via application of appropriate control inputs but the dynamics may be such that trajectories with initial conditions in those regions of the state space converge to the origin under the natural dynamics of the system. We will explore this concept more fully in the special case of linear time-invariant systems.

3.2 Reachability for Linear State Space Systems

Consider a linear, time-invariant system

$$\frac{dx}{dt} = Ax + Bu, \qquad x(0) = x_0, \tag{S3.2}$$

with $x \in \mathbb{R}^n$ and $u \in \mathbb{R}^m$, and having state transition function. In this case the state transition function is given by the convolution equation,

$$x(t) = s(t, 0, x_0, u(\,\cdot\,)) = e^{At}Bx_0 + \int_0^t e^{A(t-\tau)}Bu(\tau)\,d\tau.$$

It can be shown that if a linear system is small-time locally controllable at the origin then it is small-time locally controllable at any point x_f, and furthermore that small-time local controllability is equivalent to reachability between any two points (Exercise 3.9).

The problem of reachability for a linear time-invariant system is the same as the general case: we wish to find an input $u(\,\cdot\,)$ that can steer the system from an initial condition x_0 to a final condition x_f in a given time T. Because the system is linear, without loss of generality we can take $x_0 = 0$ (if not, replace the final position x_f with $x_f - e^{AT}x_0$. In addition, since the state space dynamics depend only on the matrices A and B, we will often state that the pair (A, B) is reachable, stabilizable, etc.

The simplest (and most commonly) used test for reachability for a linear system is to check that the reachability matrix is full rank:

$$\operatorname{rank}\begin{bmatrix} B & AB & \cdots & A^{n-1}B \end{bmatrix} = n.$$

The rank test provides a simple method for checking for reachability, but has the disadvantage that doesn't provide any quantitative insight into how "hard" it might be to either reach a given state or to assign the eigenvalues of the closed loop systems.

A better method of characterizing the reachability properties of a linear system is to make use of the fact that the system defines a linear map between the input $u(\,\cdot\,) \in \mathcal{U}$ and the state $x(T) = x_f \in \Sigma$:

$$x(T) = \mathcal{L}_T u(\,\cdot\,) = \int_0^T e^{A(T-\tau)}Bu(\tau)\,d\tau. \tag{S3.3}$$

Recall that for a linear operator in finite dimensional spaces $L : \mathbb{R}^m \to \mathbb{R}^n$ with $m > n$ that the rank of the linear operator L is the same as the rank of the linear operator $LL^* : \mathbb{R}^n \to \mathbb{R}^n$ where $L^* : \mathbb{R}^n \to \mathbb{R}^m$ is the adjoint operator (given by L^T in the case of matrices). Furthermore, if L is surjective (onto) then the least squares inverse of L is given by

$$L^+ = L^*(LL^*)^{-1} \quad \text{and} \quad LL^+ = I \in \mathbb{R}^{n\times n}.$$

More generally, the adjoint operator can be defined on a linear map between Banach spaces by defining the dual of a Banach space V to be the space V^* of continuous linear functionals on V. Given a linear function $\omega \in V^*$ we write $\langle \omega, v\rangle := \omega(v)$ to represent the application of the function ω on the element v. In the case of finite dimensional vector spaces we can associate the set V^* with V and $\langle \sigma, v\rangle$ is of the form $w^\mathsf{T} v$ where $w \in \mathbb{R}^n$.

If we have a mapping between linear spaces V and W given by $L : V \to W$, the adjoint operator $L^* : W^* \to V^*$ is defined as the unique operator that satisfies

$$\langle L^*\sigma, v\rangle = \langle \sigma, Lv\rangle \quad \text{for all } v \in V \text{ and } \sigma \in W^*.$$

Note that the application of the linear function on the left occurs in the space V and on the right occurs in the space W.

For a signal space $\mathcal{U}$, a linear functional has the form of an integral

$$\langle \omega(\,\cdot\,), u(\,\cdot\,)\rangle = \int_0^\infty \omega(\tau)\cdot u(\tau)\, d\tau$$

and so we can associate each linear function in $\mathcal{U}^*$ with a function $\omega(t)$. Given a linear mapping $\mathcal{L}_T : \mathcal{U} \to \Sigma$ of the form

$$\mathcal{L}_T(u(\,\cdot\,)) = \int_0^T h(T-\tau)u(\tau)\, d\tau$$

it can be shown that the adjoint operator ${\mathcal{L}_T}^* : \Sigma \to \mathcal{U}^*$ is given by

$$\mathcal{L}_T^*(t)w^* = \begin{cases} \langle h(T-t), w\rangle & \text{if } t \le T, \\ 0 & \text{otherwise} \end{cases}$$

where $w \in \Sigma$.

To show that a system is reachable, we need to show that $\mathcal{L}_T : \mathcal{U} \to \Sigma$ given by equation (S3.3) is full rank. Using the analysis above, the adjoint operator $\mathcal{L}_T^* : \Sigma \to \mathcal{U}^*$ is

$$(\mathcal{L}_T^* v)(t) = B^\mathsf{T} e^{A^\mathsf{T}(T-t)} v.$$

As in the finite dimensional case, the dimension of the range of the map $\mathcal{L}_T : \mathcal{U} \to \Sigma$ is the same as the dimension of the range of the map $\mathcal{L}_T\mathcal{L}_T^* : \Sigma \to \Sigma$, which is given by

$$\mathcal{L}_T\mathcal{L}_T^* = \int_0^T e^{A(T-\tau)} BB^\mathsf{T} e^{A^\mathsf{T}(T-\tau)}\, d\tau.$$

This analysis leads to the following result on reachability for a linear system. △

Theorem 3.1 (Gramian test)**.** *A pair (A, B) is reachable in time T if and only if*

$$W_c(T) = \int_0^T e^{A(T-\tau)} B B^\mathsf{T} e^{A^\mathsf{T}(T-\tau)}\, d\tau = \int_0^T e^{A\tau} B B^\mathsf{T} e^{A^\mathsf{T}\tau}\, d\tau$$

is positive definite.

The matrix $W_\mathrm{c}(T)$ provides a means to compute an input that steers a linear system from the origin to a point $x_\mathrm{f} \in \mathbb{R}^n$. Given $T > 0$, define

$$u(t) = B^\mathsf{T} e^{A^\mathsf{T}(T-t)} W_\mathrm{c}^{-1}(T) x_\mathrm{f}.$$

It follows from the definition of W_c that $x_0 \underset{T}{\rightsquigarrow} x_f$. Furthermore, it is possible to show that if the system is reachable for some $T > 0$ then it is reachable for *all* $T > 0$. Note that this computation of $u(\,\cdot\,)$ corresponds to the computation of the least squares inverse in the finite dimensional case ($u = \mathcal{L}_T^*(\mathcal{L}_T \mathcal{L}_T)^{-1} x_\mathrm{f}$).

Lemma 3.2. *If $W_c(T)$ is positive definite for some $T > 0$ then it is positive definite for all $T > 0$.*

Proof. We prove the statement by contradiction. Suppose that $W_\mathrm{c}(T)$ is positive definite for a specific $T > 0$ but that there exists $T' > 0$ such that $\operatorname{rank} W_\mathrm{c}(T') = k < n$. Then there exists a vector $v \in \mathbb{R}^n$ such that $v^\mathsf{T} W_\mathrm{c}(T') = 0$ and furthermore

$$v^\mathsf{T} W_\mathrm{c}(T') v = v^\mathsf{T} \left(\int_0^{T'} e^{A\tau} B B^\mathsf{T} e^{A^\mathsf{T}\tau} v\, d\tau \right) = 0.$$

Since the integrand is a symmetric matrix, it follows that we must have

$$v^\mathsf{T} e^{A\tau} B B^\mathsf{T} e^{A^\mathsf{T}\tau} v = 0 \quad \text{for all } \tau \le T',$$

and hence

$$\begin{aligned} v^\mathsf{T} e^{A\tau} B = 0 \quad &\Longrightarrow \quad v^\mathsf{T} B = 0 \quad \text{(evaluating at } t = 0) \\ \frac{d}{d\tau}(v^\mathsf{T} e^{A\tau} B) = v^\mathsf{T} A e^{A\tau} B = 0 \quad &\Longrightarrow \quad v^\mathsf{T} A B = 0 \\ &\vdots \\ v^\mathsf{T} A^{n-1} &B = 0. \end{aligned}$$

Therefore $v^\mathsf{T} e^{A\tau} B = 0$ for *all* τ (including $\tau > T'$) and hence $v^\mathsf{T} W_\mathrm{c}(t) = 0$ for all $t > 0$, contradicting our original hypothesis. □

If the eigenvalues of A all have negative real part, it can be shown that $W_c(t)$ converges to a constant matrix as $t \to \infty$ and we write this matrix as $W_c = W_c(\infty)$. This matrix is called the *controllability Gramian.* (Note that FBS2e uses W_r to represent the reachability *matrix* $[B\ AB\ A^2B\ \ldots]$. This is different than the controllability *Gramian.*)

Theorem 3.3. $AW_c + W_c A^\mathsf{T} = -BB^\mathsf{T}$.

Proof.

$$\begin{aligned} AW_c + W_c A^\mathsf{T} &= \int_0^\infty Ae^{A\tau}BB^\mathsf{T}e^{A^\mathsf{T}\tau}\,d\tau + \int_0^\infty e^{A\tau}BB^\mathsf{T}e^{A^\mathsf{T}\tau}A^\mathsf{T}\,d\tau \\ &= \int_0^\infty \frac{d}{dt}\left(e^{A\tau}BB^\mathsf{T}e^{A^\mathsf{T}\tau}\right)d\tau \\ &= \left(e^{At}BB^\mathsf{T}e^{A^\mathsf{T}t}\right)\Big|_{t=0}^{\infty} \\ &= 0 - BB^\mathsf{T} = -BB^\mathsf{T}. \end{aligned}$$

□

Theorem 3.4. *A linear time-invariant control system* (S3.2) *is reachable if and only if W_c is full rank and the subspace of points that are reachable from the origin is given by the image of W_c.*

Proof. Left as an exercise. Use the fact that the range of $W_c(T)$ is independent of T. □

Reachability is best captured by the Gramian since it relates directly to the map between an input vector and final state, and its norm is related to the difficulty of moving from the origin to an arbitrary state. Furthermore, the eigenvectors of W_c and the corresponding eigenvalues provide a measure of how much control effort is required to move in different directions. There are, however, several other tests for reachability that can be used for linear systems.

Theorem 3.5. *The following conditions are necessary and sufficient for reachability of a linear time-invariant system:*

- *Reachability matrix test:*

$$\text{rank}\begin{bmatrix} B & AB & \cdots & A^{n-1}B \end{bmatrix} = n.$$

- *Popov-Belman-Hautus (PBH) test:*

$$\text{rank}\left[\; sI - A \;\middle|\; B \;\right] = n$$

for all $s \in \mathbb{C}$ (suffices to check for eigenvalues of A).

Proof. (Incomplete) PBH necessity: Suppose

$$\text{rank}\left[\; \lambda I - A \;\middle|\; B \;\right] < n.$$

Then there exists $v \neq 0$ such that

$$v^\mathsf{T}\left[\; \lambda I - A \;\middle|\; B \;\right] = 0$$

and hence $v^\mathsf{T}A = \lambda v^\mathsf{T}$ and $v^\mathsf{T}B = 0$. It follows that $v^\mathsf{T}A^2 = \lambda^2 v^\mathsf{T}, \ldots, v^\mathsf{T}A^{n-1} = \lambda^{n-1}v^\mathsf{T}$ and thus

$$v^\mathsf{T}\begin{bmatrix} B & AB & \cdots & A^{n-1}B \end{bmatrix} = 0.$$

□

For both of these tests, we note that if the corresponding matrix is rank deficient, the left null space of that matrix gives directions in the state space that are unreachable (more accurately it consists of the directions in which the projected value of the state is constant along all trajectories of the system). The set of vectors orthogonal to this left null space defines a subspace V_r that represents the set of reachable states (exercise: prove this is a subspace).

Theorem 3.6. *Assume (A, B) is not reachable. Let $\operatorname{rank} W_c = r < n$. Then there exists a transformation $T \in \mathbb{R}^{n \times n}$ such that*

$$TAT^{-1} = \begin{bmatrix} A_1 & A_2 \\ 0 & A_3 \end{bmatrix}, \quad TB = \begin{bmatrix} B_1 \\ 0 \end{bmatrix},$$

where $A_1 \in \mathbb{R}^{r \times r}$, $B_1 \in \mathbb{R}^{r \times m}$, and (A_1, B_1) is reachable.

Proof. (Sketch) Let V_{r} represent the null space of W_{c} and let $\mathcal{B}_{\bar{\mathrm{r}}} = \{w_1, \ldots, w_{n-r}\}$ represent a basis for V_{r}. Complete this basis with a set of vectors $\{v_1, \ldots, v_r\}$ such that $\{v_1, \ldots, v_r, w_1, \ldots, w_{n-r}\}$ is a basis for $\mathbb{R}^n$. Use these basis vectors as the columns of the transformation T. □

We note that the null space of W_{c} is uniquely defined, though the basis for that space is not unique. This subspace represents the set of linear functions on the state space whose values are constant and hence provides a characterization of the unreachable states of the system. The complement of that space is not a subspace, although if we look at the points that are reachable from the origin, this does form a subspace. We will return to this point in more detail when we discuss the Kalman decomposition in Chapter 5. △

Finally, we note that a system that is reachable can be written in *reachable canonical form* (see FBS2e). This is primarily useful for proofs.

3.3 System Norms

Consider a stable state space system with no direct term and with system matrices A, B, and C. Let W_{c} be the controllability Gramian for the system and let $G(t)$ represent the impulse response function for the system and $\hat{G}(s)$ represent the corresponding transfer function (Laplace transform of the impulse response). Recall that the 2-norm to ∞-norm gain for a linear input/output system is given by $\|\hat{G}\|_2$.

Theorem 3.7. $\|\hat{G}\|_2 = \sqrt{CW_cC^\mathsf{T}}$.

Proof. The impulse response function given by

$$\begin{aligned} G(t) = Cx_\delta(t) &= C \int_0^t A^{A(t-\tau)} B\delta(\tau)\, d\tau \\ &= Ce^{At}B, \qquad t > 0. \end{aligned}$$

The system norm is given by

$$\begin{aligned} \|\hat{G}\|_2^2 &= \|G\|_2^2 \\ &= \int_0^\infty \big(Ce^{A\tau}B\big)\big(B^\mathsf{T} e^{A^\mathsf{T}\tau} C^\mathsf{T}\big)\, d\tau \\ &= C \left(\int_0^\infty e^{At} B B^\mathsf{T} e^{A^\mathsf{T}\tau}\, d\tau \right) C^\mathsf{T} \\ &= CW_cC^\mathsf{T}. \end{aligned}$$

□

The more common norm in control system design is the 2-norm to 2-norm system gain, which is given by $\|\hat{G}\|_\infty$. To compute the ∞-norm of a transfer function, we define

$$H_\gamma = \begin{bmatrix} A & \frac{1}{\gamma}BB^\mathsf{T} \\ -C^\mathsf{T}C & -A^\mathsf{T} \end{bmatrix} \in \mathbb{R}^{2n\times 2n}.$$

The system gain can be determined in terms of H_γ as follows.

Theorem 3.8. *$\|\hat{G}\|_\infty < \gamma$ is an only if H_γ has no eigenvalues on the $j\omega$ axis.*

Proof. DGKF. □

To numerically compute the H_∞ norm, we can use the bisection method to determine γ to arbitrary accuracy.

3.4 Stabilization via Linear Feedback

We now consider the problem of stabilization, as defined in Definition 3.4. For a linear system, we will consider feedback laws of the form $u = -Kx$ (the negative sign is a convention associated with the use of "negative" feedback), so that

$$\frac{dx}{dt} = Ax + Bu = (A - BK)x.$$

One of the goals of introducing negative feedback is to stabilize an otherwise unstable system at the origin. In addition, state feedback can be used to "design the dynamics" of the close loop system by attempting to assign the eigenvalues of the closed loop system to specific values.

Theorem 7.3 states that if a (single-input) system is reachable then it is possible to assign the eigenvalues of the closed loop system to arbitrary values. This turns out to be true for the multi-input case as well and is proved in a similar manner (by using an appropriate normal form).

Using the decomposition theorem 3.6 it is easy to see that the question of stabilizability for a linear system comes down to the question of whether the dynamics in the unreachable space ($\dot{z} = A_3 z$) are stable, since these eigenvalues cannot be changed through the use of state feedback.

Although eigenvalue placement provides an easy method for designing the dynamics of the closed loop system, it is rarely used directly since it does not provide any guidelines for trading off the size of the inputs required to stabilize the dynamics versus the properties of the closed loop response. This is explored in a bit more detail in FBS2e Section 14.6 (Robust Pole Placement).

3.5 Exercises

3.1 (Sontag 3.1.2/3.1.3) Prove the following statements:

(a) If $(x,\sigma) \rightsquigarrow (z,\tau)$ and $(z,\tau) \rightsquigarrow (y,\mu)$, then $(x,\sigma) \rightsquigarrow (y,\mu)$.

(b) If $(x,\sigma) \rightsquigarrow (y,\mu)$ and if $\sigma < \tau < \mu$, then there exists a $z \in \mathcal{X}$ such that $(x,\sigma) \rightsquigarrow (z,\tau)$ and $(z,\tau) \rightsquigarrow (y,\mu)$.

(c) If $x \underset{T}{\rightsquigarrow} y$ for some $T > 0$ and if $0 < t < T$, then there is some $z \in \mathcal{X}$ such that $x \underset{t}{\rightsquigarrow} z$ and $z \underset{T-t}{\rightsquigarrow} y$.

(d) If $x \underset{t}{\rightsquigarrow} z$, $z \underset{s}{\rightsquigarrow} y$, and Σ is time-invariant, then $x \underset{t+s}{\rightsquigarrow} y$.

(e) If $x \rightsquigarrow z$, $z \rightsquigarrow y$, and Σ is time-invariant, then $x \rightsquigarrow y$.

(f) Given examples that show that properties (d) and (e) may be false if Σ is not time-invariant.

(g) Even for time-invariant systems, it is not necessarily true that $x \rightsquigarrow z$ implies that $z \rightsquigarrow x$ (so, "$\rightsquigarrow$" is not an equivalence relation).

3.2 (FBS2e 7.1) Consider the double integrator. Find a piecewise constant control strategy that drives the system from the origin to the state $x = (1, 1)$.

3.3 (FBS2e 7.2) Extend the argument in Section 7.1 in *Feedback Systems* to show that if a system is reachable from an initial state of zero, it is reachable from a nonzero initial state.

3.4 (FBS2e 7.3) Consider a system with the state x and z described by the equations

$$\frac{dx}{dt} = Ax + Bu, \qquad \frac{dz}{dt} = Az + Bu.$$

If $x(0) = z(0)$ it follows that $x(t) = z(t)$ for all t regardless of the input that is applied. Show that this violates the definition of reachability and further show that the reachability matrix W_{r} is not full rank. What is the rank of the reachability matrix?

3.5 (FBS2e 7.6) Show that the characteristic polynomial for a system in reachable canonical form is given by equation (7.7) and that

$$\frac{d^n z_k}{dt^n} + a_1 \frac{d^{n-1} z_k}{dt^{n-1}} + \cdots + a_{n-1} \frac{dz_k}{dt} + a_n z_k = \frac{d^{n-k} u}{dt^{n-k}},$$

where z_k is the kth state.

3.6 (FBS2e 7.7) Consider a system in reachable canonical form. Show that the inverse of the reachability matrix is given by

$$\tilde{W}_{\mathrm{r}}^{-1} = \begin{bmatrix} 1 & a_1 & a_2 & \cdots & a_{n-1} \\ & 1 & a_1 & \cdots & a_{n-2} \\ & & 1 & \ddots & \vdots \\ & 0 & & \ddots & a_1 \\ & & & & 1 \end{bmatrix}.$$

3.7 (FBS2e 7.10) Consider the system

$$\frac{dx}{dt} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u, \qquad y = \begin{bmatrix} 1 & 0 \end{bmatrix} x,$$

with the control law

$$u = -k_1 x_1 - k_2 x_2 + k_{\mathrm{f}} r.$$

Compute the rank of the reachability matrix for the system and show that eigenvalues of the system cannot be assigned to arbitrary values.

3.8 (FBS2e 7.11) Let $A \in \mathbb{R}^{n \times n}$ be a matrix with characteristic polynomial $\lambda(s) = \det(sI - A) = s^n + a_1 s^{n-1} + \cdots + a_{n-1}s + a_n$. Show that the matrix A satisfies

$$\lambda(A) = A^n + a_1 A^{n-1} + \cdots + a_{n-1}A + a_n I = 0,$$

where the zero on the right hand side represents a matrix of elements with all zeros. Use this result to show that A^n can be written in terms of lower order powers of A and hence any matrix polynomial in A can be rewritten using terms of order at most $n - 1$.

3.9 Show that for a linear time-invariant system, the following notions of controllability are equivalent:

(a) Reachability to the origin ($x_0 \rightsquigarrow 0$).

(b) Reachability from the origin ($0 \rightsquigarrow x_\mathrm{f}$).

(c) Small-time local controllability ($x_0 \rightsquigarrow B(x_0, \epsilon)$).

3.10 (Sontag 3.3.4) Assume that the pair (A, B) is not controllable with $\dim R(A, B) = \operatorname{rank} W_\mathrm{c} = r < n$. From Lemma 3.3.3, there exists an invertible matrix $T \in \mathbb{R}^{n \times n}$ such that the matrices $\tilde{A} := T^{-1}AT$ and $\tilde{B} := T^{-1}B$ have the block structure

$$\tilde{A} = \begin{bmatrix} A_1 & A_2 \\ 0 & A_3 \end{bmatrix}, \qquad \tilde{B} = \begin{bmatrix} B_1 \\ 0 \end{bmatrix},$$

where $A_1 \in \mathbb{R}^{r \times r}$ and $B_1 \in \mathbb{R}^{r \times m}$. Prove that (A_1, B_1) is itself a controllable pair.

3.11 (Sontag 3.3.6) Prove that if

$$A = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 & 0 \\ 0 & \lambda_2 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \lambda_{n-1} & 0 \\ 0 & 0 & \cdots & 0 & \lambda_n \end{bmatrix}, \qquad B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_{n-1} \\ b_n \end{bmatrix}$$

then (A, B) is controllable if and only if $\lambda_i \neq \lambda_j$ for each $i \neq j$ and all $b_i \neq 0$.

3.12 (Sontag 3.3.14) Let (A, B) correspond to a time-invariant *discrete-time* linear system Σ. Recall that null-controllability means that every state can be controlled to zero. Prove that the following conditions are equivalent:

(a) Σ is null-controllable.

(b) The image of A^n is contained in the image of $R(A, B)$.

(c) In the decomposition in Sontag, Lemma 3.3.3, A_3 is nilpotent.

(d) $\operatorname{rank}[\lambda I - A, B] = n$ for all nonzero $\lambda \in \tilde{\mathbb{R}}$.

(e) $\operatorname{rank}[\lambda I - A, B] = n$ for all $\lambda \in \tilde{\mathbb{R}}$.