Feedback Systems:
Notes on Linear Systems Theory

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These notes are a supplement for the second edition of *Feedback Systems* by Åström and Murray (referred to as FBS2e), focused on providing some additional mathematical background and theory for the study of linear systems.

Chapter 2

Linear Input/Output Systems

2.1 Matrix Exponential

Let $x(t) \in \mathbb{R}^n$ represent that state of a system whose dynamics satisfy the linear differential equation

$$\frac{d}{dt}x(t) = Ax(t), \qquad A \in \mathbb{R}^{n\times n},\ t \in [0, \infty).$$

The *initial value problem* is to find $x(t)$ given $x(0)$. The approach that we take is to show that there is a unique solution of the form $x(t) = e^{At}x(0)$ and then determine the properties of the solution (e.g., stability) as a function of the properties of the matrix A.

Definition 2.1. Let $S \in \mathbb{R}^{n\times n}$ be a square matrix. The *matrix exponential* of S is given by

$$e^S = I + S + \frac{1}{2}S^2 + \frac{1}{3!}S^2 + \cdots + \frac{1}{k!}S^k + \ldots$$

Proposition 2.1. *The series* $\sum_{k=0}^{\infty} \frac{1}{k!}S^k$ *converges for all* $S \in \mathbb{R}^{n\times n}$.

Proof. Simple case: Suppose S has a basis of eigenvectors $\{v_1, \ldots, v_n\}$. Then

$$\begin{aligned} e^S v_i &= (I + S + \cdots + \frac{1}{k!}S^k + \ldots)v_i \\ &= (1 + \lambda_i + \cdots + \frac{1}{k!}\lambda_i^k + \ldots)v_i \\ &= e^{\lambda_i}v_i, \end{aligned}$$

which implies that $e^S x$ is well defined and finite (since this is true for all basis elements.

General case: Let $\|S\| = a$. Then

$$\|\frac{1}{k!}S^k\| \le \frac{1}{k!}\|S\|^k = \frac{a^k}{k!}.$$

Hence

$$\|e^S\| \le \sum_{k=1}^{\infty} \frac{a^k}{k!} = e^a = e^{\|S\|}$$

and so $e^S x$ is well-defined and finite. □

Proposition 2.2. *If $P, T \in \mathbb{R}^n \to \mathbb{R}^n$ and $S = PTP^{-1}$ then*

$$e^S = e^{PTP^{-1}} = Pe^T P^{-1}.$$

Proof.

$$\begin{aligned} e^S &= \sum \frac{1}{k!} S^k = \sum \frac{1}{k!} (PTP^{-1})^k \\ &= \dots \frac{1}{k!} (PTP^{-1}) \cdot (PTP^{-1}) \cdots (PTP^{-1}) \cdots \\ &= \sum \frac{1}{k!} PT^k P^{-1}) = P\Big(\sum \frac{T^k}{k!}\Big) P^{-1}) = Pe^T P^{-1}. \end{aligned}$$

□

Proposition 2.3. *If $S, T : \mathbb{R}^n \to \mathbb{R}^n$ commute ($ST = TS$) then $e^{S+T} = e^S e^T$.*

Proof. (basic idea) The first few terms of expansion for the matrix exponential are given by

$$\begin{aligned} (S+T)^0 &= I \\ (S+T)^1 &= S + T \\ (S+T)^2 &= (S+T)(S+T) = S^2 + ST + TS + T^2 \\ &= S^2 + 2ST + T^2 \quad \text{only if ST = TS!} \\ (S+T)^0 &= (S+T)(S+T)(S+T) \\ &= S^3 + S^2T + STS + ST^2 + TS^2 + TST + T^2S + T^3 \\ &= S^3 + 3S^2T + 3ST^2 + T^2 \quad \text{only if ST = TS!.} \end{aligned}$$

The general form becomes

$$(S+T)^k = \underbrace{\sum_{i=1}^{k} \binom{k}{i} S^i T^{k-i}}_{\text{binomial theorem}} = \sum_{i=1}^{k} \frac{k!}{i!(k-i)!} S^i T^{k-i}.$$

□

2.2 Convolution Equation

We now extend our results to include an input. Consider the non-autonomous differential equation

$$\dot{x} = Ax + b(t), \qquad x(0) = x_0. \tag{S2.1}$$

Theorem 2.4. *If $b(t)$ is a (piecewise) continuous signal, then there is a unique $x(t)$ satisfying equation* (S2.1) *given by*

$$x(t) = e^{At} x_0 + \int_0^t e^{T(t-\tau)} b(\tau)\, d\tau.$$

Proof. (existence only) Note that $x(0) = x_0$ and

$$\begin{aligned}
\frac{d}{dt}x(t) &= Ax(t) + \frac{d}{dt}\left(e^{At}\int_0^t e^{-A\tau}b(\tau)\,d\tau\right) \\
&= Ax(t) = Ae^{At}\left(\int_0^t e^{-A\tau}b(\tau)\,d\tau\right) + e^{At}\left(e^{-At}b(t)\right) \\
&= \left[Ax(t) + A\int_0^t e^{A(t-\tau)}b(\tau)\,d\tau\right] + b(t) \\
&= Ax(t) + b(t).
\end{aligned}$$

□

Note that the form of the solution is a combination of the initial condition response ($e^{At}x_0$) and the forced response ($\int_0^t \dots$). Linearity in the initial condition and the input follows from linearity of matrix multiplication and integration.

An alternative form of the solution can be obtained by defining the *fundamental matrix* $\Phi(t) = e^{At}$ as the solution of the matrix differential equation

$$\dot{\Phi} = A\Phi, \quad \Phi(0) = I.$$

Then the solution can be written as

$$x(t) = \Phi(t)x_0 + \underbrace{\int_0^t \Phi(t-\tau)b(\tau)\,d\tau}_{\textit{convolution}\text{ of }\Phi\text{ and }b(t)}.$$

Φ thus acts as a Green's function.

A common situation is that $b(t) = B \cdot a\sin(\omega t)$ where $B \in \mathbb{R}^n$ is a vector and $a\sin(\omega t)$ is a sinusoid with amplitude a and frequency ω. In addition, we wish to consider a specific combination of states $y = Cx$, where $C : \mathbb{R}^n \to \mathbb{R}$:

$$\begin{aligned}
\dot{x}(t) &= Ax + Bu(t) & u(t) &= a\sin(\omega t) \\
y(t) &= Cx & x(0) &= x_0.
\end{aligned} \tag{S2.2}$$

Theorem 2.5. *Let $H(s) = C(sI - A)^{-1}B$ and define $M = |H(i\omega)|$, $\phi = \arg H(i\omega)$. Then the sinusoidal response for the system in equation* (S2.2) *is given by*

$$y(t) = Ce^{At}x(0) + aM\sin(\omega t + \phi).$$

A proof can be found in FBS or worked out by using $\sin(\omega t) = \frac{1}{2}(e^{i\omega t} - e^{-i\omega t})$. The function $H(i\omega)$ gives the *frequency response* for the linear system. The function $H : \mathbb{C} \to \mathbb{C}$ is called the *transfer function* for the system.

2.3 Linear System Subspaces

To study the properties of a linear dynamical system, we study the properties of the eigenvalues and eigenvectors of the dynamics matrix $A \in \mathbb{R}^{n \times n}$. We will make use of the *Jordan canonical form* for a matrix. Recall that given any matrix $A \in \mathbb{R}^{n\times n}$ there exists a transformation $T \in \mathbb{C}^{n\times n}$ such that

$$J = TAT^{-1} = \begin{bmatrix} J_1 & & 0 \\ & \ddots & \\ 0 & & J_N \end{bmatrix}, \qquad J_k = \begin{bmatrix} \lambda_k & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & \lambda_k \end{bmatrix} \in \mathbb{R}^{m_k \times m_k}.$$

This is the complex version of the Jordan form. There is also a real version with $T \in \mathbb{R}^{n\times n}$ in which case the Jordan blocks representing complex eigenvalues have the form

$$J_k = \left[\begin{array}{c|c|c|c} \begin{matrix} a_k & -b_k \\ b_k & a_k \end{matrix} & \begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix} & 0 & 0 \\ \hline 0 & \ddots & \ddots & 0 \\ \hline 0 & 0 & \ddots & \begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix} \\ \hline 0 & 0 & 0 & \begin{matrix} a_k & -b_k \\ b_k & a_k \end{matrix} \end{array}\right]$$

In both the real and complex cases the transformation matrices T consist of a set of generalized eigenvectors $w_{k_1}, \dots, w_{k_{m_k}}$ corresponding to the eigenvalue λ_k.

Returning now to the dynamics of a linear system, let $A \in \mathbb{R}^{n\times n}$ be a square matrix representing the dynamics matrix with eigenvalues $\lambda_j = a_j + ib_j$ and corresponding (generalized) eigenvectors $w_j = u_j + iv_j$ (with $v_j = 0$ if $b_j = 0$). Let $\mathcal{B}$ be a basis of $\mathbb{R}^n$ given by

$$\mathcal{B} = \{\underbrace{u_1, \dots, u_p,}_{\text{real } \lambda_j} \underbrace{u_{p+1}, v_{p+1}, \dots, u_{p+q}, v_{p+q}}_{\text{complex } \lambda_j}\}. \tag{S2.3}$$

Definition 2.2. Given $A \in \mathbb{R}^n$ and basis vector $\mathcal{B}$ as in equation (S2.3), define

1. *Stable subspace*: $E^s = \operatorname{span}\{u_j, v_j : a_j < 0\}$;
2. *Unstable subspace*: $E^u = \operatorname{span}\{u_j, v_j : a_j > 0\}$;
3. *Center subspace*: $E^c = \operatorname{span}\{u_j, v_j : a_j = 0\}$.

These three subspaces can be used to characterize the behavior an unforced linear system. Since $E^s \cap E^u = \{0\}$, $E^s \cap E^c = \{0\}$, and $E^c \cap E^u = \{0\}$, it follows that any vector x can be written as a unique decomposition

$$x = u + v + w, \qquad u \in E^s,\ v \in E^c,\ w \in E^u,$$

and thus $\mathbb{R}^n = E^s \oplus E^c \oplus E^u$ where $\oplus$ is the direct sum of two linear subspaces, defined as $S_1 \oplus S_2 = \{u + v : u \in S_1,\ v \in S_2\}$. If all eigenvalues of A have nonzero real part, so that $E^c = \{0\}$ then the linear system $\dot{x} = Ax$ is said to be *hyperbolic*.

Definition 2.3. A subspace $E \subset \mathbb{R}^n$ is *invariant* with represent to the matrix $A \in \mathbb{R}^{n\times n}$ if $AE \subset E$ and is *invariant with respect to the flow* $e^{At} : \mathbb{R}^n \to \mathbb{R}^n$ if $e^{At}E \subset E$ for all t.

Proposition 2.6. *Let E be the generalized eigenspace of A corresponding to an eigenvalue λ. Then $AE \subset E$.*

Proof. Let $\{v_1, \ldots, v_k\}$ be a basis for the generalized eigenspace of A. Then for every $v \in E$ we can write

$$v = \sum \alpha_j v_j \qquad \Longrightarrow \qquad Av = \sum \alpha_j A v_j$$

where α_j is the eigenvalue for the generalized eigenspace. Since $v_1, \ldots, v_k$ space the generalized eigenvectors, we know that $(\lambda I - A)_j^k v_j = 0$ for some minimal k_j associated with v_j. It follows that

$$(A - \lambda I)v_j \in \ker(A - \lambda I)^{k_j - 1} \subset E$$

and hence $Av_j = w + \alpha_j v_j$ where $w \in E$, which implies that $Av_j \in E$. □

Proposition 2.7. *The subspaces E^s, E^c, and E^u are all invariant under A and e^{At}.*

Proof. Invariance under A follows from Proposition 2.6. To show invariance of the flow note that

$$e^{At} = (I + At + \frac{1}{2}A^2t^2 + \ldots)$$

so

$$e^{At}E \subset E \oplus AE \oplus A^2E \oplus \cdots \subset E.$$

□

Theorem 2.8 (Stability of linear systems)**.** *The following statements are equivalent*

1. *$E^s = \mathbb{R}^n$ (i.e., all eigenvalues have negative real part);*
2. *For all $x_0 \in \mathbb{R}^n$, $\lim_{t\to\infty} e^{At}x_0 = 0$ (trajectories converge to the origin);*
3. *There exist constants $a, c, m, M > 0$ such that*

$$me^{-at}\|x_0\| \leq \|e^{At}x_0\| \leq Me^{-ct}\|x_0\|$$

(exponential rate of convergence).

Proof. To show the equivalence of (1) and (2) we assume without loss of generality that the matrix is transformed into (real) Jordan canonical form. It can be shown that each Jordan block J_k can be decomposed into a diagonal matrix $S_k = \lambda_k I$ and a nilpotent matrix N_k consisting of 1's on the superdiagonal. The properties of the decomposition $J_k = S_k + N_k$ are that S_k and N_k commute and $N_k^{m_k} = 0$ (so that N_k is *nilpotent*). From these two properties we have that

$$e^{J_k t} = e^{\lambda_k I t} e^{N_k t} = e^{\lambda_k t}(I + N_k + \frac{1}{2}N^2t^2 + \cdots + \frac{1}{(m_k - 1)!}N^{m_k - 1}t^{m_k - 1}).$$

A similar decomposition is possible for complex eigenvalues, with the diagonal elements of $e^{S_k t}$ taking the form

$$e^{a_k t}\begin{bmatrix}\cos(b_k t) & -\sin(b_k t) \\ \sin(b_k t) & \cos(b_k t)\end{bmatrix}$$

and the matrix N_k being a block matrix with superdiagonal elements given by the 2×2 identity matrix.

For the real blocks we have $\lambda_k < 0$ and for the complex blocks we have the $a_k < 0$ and it follows that $e^{Jt} \to 0$ as $t \to \infty$ (making use of the fact that $e^{\lambda t} t^m \to 0$ for any $\lambda > 0$ and $m \geq 0$). It follows that (1) and (2) are thus equivalent.

To show (3) we need two additional facts, which we state without proof.

Lemma 2.9. *Let $T \in \mathbb{R}^{n \times n}$ be an invertible transformation and let $y = Tx$. Then there exists constants m and M such that*

$$m\|x\| \leq \|y\| \leq M\|x\|.$$

Lemma 2.10. *Let $A \in \mathbb{R}^{n \times n}$ and assume $\alpha < |\mathrm{Re}\,(\lambda)| < \beta$ for all eigenvalues λ. Then there exists a set of coordinates $y = Tx$ such that*

$$\alpha\|y\|^2 \leq y^T(TAT^{-1})y \leq \beta\|y\|^2.$$

Using these two lemmas we can account for the transformation in converting the system into Jordan canonical form. The only remaining element to prove is that a function of the form $h(t) = e^{\lambda_k t} t^m < \gamma e^{\lambda t}$ for some λ and $\gamma > 0$. This follows from the fact that a function of the form $e^{-\epsilon t} t^m l$ is continuous and zero at $t = 0$ and at $t = \infty$ and thus $e^{-\epsilon t} t^m l$ is bounded above and below. From this we can show (with a bit more work) that for any Jordan block J_k there exists $\gamma > 0$ and $\lambda_k < \lambda < 0$ such that $me^{-at} < \|e^{J_k t} x_0\| < Me^{-ct}$ where $a < \lambda_k < c < 0$. The full result follows by combining all of the various bounds. □

A number of other stability results can be derived along the same lines as the arguments above. For example, if $\mathbb{R}^n = E^u$ (all eigenvalues have positive real part) then all solutions to the initial value problem diverge, exponentially fast. If $\mathbb{R}^n = E^u \oplus E^s$ then we have a mixture of stable and unstable spaces. Any initial condition with a component in E^u diverges, but if $x_0 \in E^s$ then the solution converges to zero.

The unresolved case is when $E^c \neq \{0\}$. In this case, the solutions corresponding to this subspace will have the form

$$(I + Nt + \frac{1}{2}N^2t^2 + \cdots + \frac{1}{k!}N^k t^k$$

for real eigenvalues and

$$\begin{bmatrix} \cos(b_k t) & -\sin(b_k t) \\ \sin(b_k t) & \cos(b_k t) \end{bmatrix} (I + Nt + \frac{1}{2}N^2t^2 + \cdots + \frac{1}{k!}N^k t^k$$

for complex eigenvalues. Convergence in this subspace depends on N. If $N = 0$ then the solutions remain bounded but do not converge to the original (stable in the sense of Lyapunov). If $N \neq 0$ the solutions diverge, but closer than the exponential case. The case of a nonlinear system whose linearization as a non-trivial center subspace leads to a center "manifold" for the nonlinear system and stability depends on the nonlinear characteristics of the system.

2.4 Input/output stability

A system is called bounded input/bounded output (BIBO) stable if a bounded input gives a bounded output for all initial states. A system is called input to state stable (ISS) if $\|x(t)\| \leq \beta(\|x(0)\|) + \gamma(\|u\|)$ where β and γ are monotonically increasing functions that vanish at the origin.

2.5 Time-Varying Systems

Suppose that we have a time-varying ("non-autonomous"), nonhomogeneous linear system with dynamics of the form

$$\begin{aligned} \frac{dx}{dt} &= A(t)x + b(t), \\ y &= C(t)x + d(t), \end{aligned} \qquad x(0) = x_0. \tag{S2.4}$$

A matrix $\Phi(t,s) \in \mathbb{R}^{n\times n}$ is called the *fundamental matrix* for $\dot{x} = A(t)x$ if

1. $\frac{d}{dt}\Phi(t,s) = A(t)\Phi(t,s)$ for all s;
2. $\Phi(s,s) = I$;
3. $\det \Phi(t,s) \neq$ for all s, t.

Proposition 2.11. *If $\Phi(t,s)$ exists for $\dot{x} = A(t)x$ then the solution to equation* (S2.4) *is given by*

$$x(t) = \Phi(t,0)x_0 + \int_0^t \Phi(t,\tau)b(\tau)\,d\tau.$$

This solution generalizes the solution for linear time-invariant systems and we see that the structure of the solution—an initial condition response combined with a convolution integral—is preserved. If $A(t) = A$ is a constant, then $\Phi(t,s) = e^{A(t-s)}$ and we recover our previous solution. The matrix $\Phi(t,s)$ is also called the *state transition matrix* since $x(t) = \Phi(t,s)x(s)$ and $\Phi(t,\tau)\Phi(\tau,s) = \Phi(t,s)$ for all $t > \tau > s$. Solutions for $\Phi(t,s)$ exists for many different time-varying systems, including periodic systems, systems that are sufficiently smooth and bounded, etc.

Example 2.1. Let

$$A(t) = \begin{bmatrix} -1 & e^{at} \\ 0 & -1 \end{bmatrix}.$$

To find $\Phi(t,s)$ we have to solve the matrix differential equation

$$\begin{bmatrix} \dot{\Phi}_{11} & \dot{\Phi}_{12} \\ \dot{\phi}_{21} & \dot{\Phi}_{22} \end{bmatrix} = \begin{bmatrix} -1 & e^{at} \\ 0 & -1 \end{bmatrix} \begin{bmatrix} \Phi_{11} & \Phi_{12} \\ \phi_{21} & \Phi_{22} \end{bmatrix},$$

with $\Phi(s,s) = I$, which serves as an initial condition for the system. We can break the matrix equation into its individual elements. Beginning with the equations for the bottom row of the matrix, we have

$$\begin{aligned} \dot{\Phi}_{21} &= -\Phi_{21} \qquad \Longrightarrow \qquad \Phi_{21}(t,s) = e^{-(t-s)}\Phi_{21}(s,s) = 0, \\ \dot{\Phi}_{22} &= -\Phi_{22} \qquad \Longrightarrow \qquad \Phi_{22}(t,s) = e^{-(t-s)}\Phi_{22}(s,s) = e^{-(t-s)}. \end{aligned}$$

Now use Φ_{21} and Φ_{22} to solve for Φ_{11} and Φ_{12}:

$$\begin{aligned} &\dot{\Phi}_{11} = -\Phi_{11} \qquad \Longrightarrow \qquad \Phi_{11}(t,s) = e^{-(t-s)} \\ &\dot{\Phi}_{12} = -\Phi_{12} + e^{at}e^{-(t-s)} = -\Phi_{12}(t,s) = e^{(a-1)t+s}. \end{aligned}$$

This last equation is of the form $\dot{x} = -x + b(t)$ and so we can solve it using the solution for a linear differential equation:

$$\begin{aligned}
\Phi_{12}(t,s) &= e^{-(t-s)}\Phi_{12}(s,s) + \int_s^t e^{-(t-\tau)} e^{(a-1)\tau+s}\, d\tau \\
&= e^{-(t-s)} \int_s^t e^{a\tau}\, d\tau = e^{-(t-s)} \left(\frac{1}{a}e^{a\tau}\right)\Bigg|_s^t \\
&= e^{-(t-s)} \left(\frac{1}{a}e^{at} - \frac{1}{a}e^{as}\right) \\
&= \frac{1}{a}e^{at-(t-s)} - \frac{1}{a}e^{as-(t-s)}.
\end{aligned}$$

Combining all of the elements, the fundamental matrix is thus given by

$$\Phi(t,s) = \begin{bmatrix} e^{-(t-s)} & \frac{1}{a}\left(e^{at-(t-s)} - e^{as-(t-s)}\right) \\ 0 & e^{-(t-s)} \end{bmatrix}.$$

The properties of the fundamental matrix can be verified by direct calculation (and are left to the reader).

The solution for the unforced system ($b(t) = 0$) is given by

$$x(t) = \Phi(t,0)x(0) = \begin{bmatrix} e^{-t} & \frac{1}{a}\left(e^{(a-1)t} - e^{-t}\right) \\ 0 & e^{-t} \end{bmatrix} x(0).$$

We see that although the eigenvalues of $A(t)$ are both -1, if $a > 1$ then some solutions of the differential equation diverge. This is an example that illustrates that for a linear system $\dot{x} = A(t)x$ stability requires more than $\mathrm{Re}\,(\lambda_A) < 0$.

A common situation is one in which $A(t)$ is period with period T:

$$\frac{dx}{dt} = A(t)x(t), \qquad A(t+T) = A(t).$$

In this case, we can show that the fundamental matrix has the form

$$\Phi(t+T,s) = \Phi(T,0)\Phi(t,s).$$

This property allows us to compute the fundamental matrix just over the period $[0,T]$ (e.g., numerically) and use this to determine the fundamental matrix at any future time. Additional explotation of the structure of the problem is also possible, as the next theorem illustrates.

Theorem 2.12 (Floquet). *Let $A(t)$ be piecewise continuous and T-periodic. Define $P(t) \in \mathbb{R}^{n\times n}$ as*

$$P(t) = \Phi(t,0)e^{-Bt}, \qquad B = \Phi(T,0).$$

Then

1. *$P(t+T) = P(t)$;*
2. *$P(0) = I$ and $\det P(t) \neq 0$;*

3. $\Phi(t,s) = P(t)e^{B(t-s)}P^{-1}(s)$;

4. If we set $z(t) = P^{-1}(t)x(t)$ *then* $\dot{z} = Bz$.

A consequence of this theorem is that in "rotating" ccoordinates z we can determine the stability properties of the system by examination of the matrix B. In particular, if $z(t) \to 0$ then $x(t) \to 0$. For a proof, see Callier and Desoer [3].

2.6 Exercises

2.1 (FBS2e 6.1) Show that if $y(t)$ is the output of a linear time-invariant system corresponding to input $u(t)$, then the output corresponding to an input $\dot{u}(t)$ is given by $\dot{y}(t)$. (Hint: Use the definition of the derivative: $\dot{z}(t) = \lim_{\epsilon \to 0}\bigl(z(t+\epsilon) - z(t)\bigr)/\epsilon$.)

2.2 (FBS2e 6.2) Show that a signal $u(t)$ can be decomposed in terms of the impulse function $\delta(t)$ as

$$u(t) = \int_0^t \delta(t-\tau)u(\tau)\,d\tau$$

and use this decomposition plus the principle of superposition to show that the response of a linear, time-invariant system to an input $u(t)$ (assuming a zero initial condition) can be written as a convolution equation

$$y(t) = \int_0^t h(t-\tau)u(\tau)\,d\tau,$$

where $h(t)$ is the impulse response of the system. (Hint: Use the definition of the Riemann integral.)

2.3 (FBS2e 6.4) Assume that $\zeta < 1$ and let $\omega_{\text{d}} = \omega_0\sqrt{1-\zeta^2}$. Show that

$$\exp\begin{bmatrix} -\zeta\omega_0 & \omega_{\text{d}} \\ -\omega_{\text{d}} & -\zeta\omega_0 \end{bmatrix} t = e^{-\zeta\omega_0 t}\begin{bmatrix} \cos\omega_{\text{d}}t & \sin\omega_{\text{d}}t \\ -\sin\omega_{\text{d}}t & \cos\omega_{\text{d}}t \end{bmatrix}.$$

Also show that

$$\exp\left(\begin{bmatrix} -\omega_0 & \omega_0 \\ 0 & -\omega_0 \end{bmatrix} t\right) = e^{-\omega_0 t}\begin{bmatrix} 1 & \omega_0 t \\ 0 & 1 \end{bmatrix}.$$

Use the results of this problem and the convolution equation to compute the unit step response for a spring mass system

$$m\ddot{q} + c\dot{q} + kq = F$$

with initial condition $x(0)$.

2.4 (FBS2e 6.6) Consider a linear system with a Jordan form that is non-diagonal.

(a) Prove Proposition 6.3 in *Feedback Systems* by showing that if the system contains a real eigenvalue $\lambda = 0$ with a nontrivial Jordan block, then there exists an initial condition with a solution that grows in time.

(b) Extend this argument to the case of complex eigenvalues with $\text{Re}\,\lambda = 0$ by using the block Jordan form

$$J_i = \begin{bmatrix} 0 & \omega & 1 & 0 \\ -\omega & 0 & 0 & 1 \\ 0 & 0 & 0 & \omega \\ 0 & 0 & -\omega & 0 \end{bmatrix}.$$

2.5 (FBS2e 6.8) Consider a linear discrete-time system of the form

$$x[k+1] = Ax[k] + Bu[k], \qquad y[k] \quad = Cx[k] + Du[k].$$

(a) Show that the general form of the output of a discrete-time linear system is given by the discrete-time convolution equation:

$$y[k] = CA^k x[0] + \sum_{j=0}^{k-1} CA^{k-j-1} Bu[j] + Du[k].$$

(b) Show that a discrete-time linear system is asymptotically stable if and only if all the eigenvalues of A have a magnitude strictly less than 1.

(c) Show that a discrete-time linear system is unstable if any of the eigenvalues of A have magnitude greater than 1.

(d) Derive conditions for stability of a discrete-time linear system having one or more eigenvalues with magnitude identically equal to 1. (Hint: Use Jordan form.)

(e) Let $u[k] = \sin(\omega k)$ represent an oscillatory input with frequency $\omega < \pi$ (to avoid "aliasing"). Show that the steady-state component of the response has gain M and phase θ, where

$$Me^{i\theta} = C(e^{i\omega} I - A)^{-1} B + D.$$

(f) Show that if we have a nonlinear discrete-time system

$$\begin{aligned} x[k+1] &= f(x[k], u[k]), \qquad && x[k] \in \mathbb{R}^n,\ u \in \mathbb{R}, \\ y[k] &= h(x[k], u[k]), \qquad && y \in \mathbb{R}, \end{aligned}$$

then we can linearize the system around an equilibrium point (x_e, u_e) by defining the matrices A, B, C, and D as in equation (6.35).

2.6 Using the computation for the matrix exponential, show that equation (6.11) in *Feedback Systems* holds for the case of a 3×3 Jordan block. (Hint: Decompose the matrix into the form $S + N$, where S is a diagonal matrix.)

2.7 Consider a stable linear time-invariant system. Assume that the system is initially at rest and let the input be $u = \sin \omega t$, where ω is much larger than the magnitudes of the eigenvalues of the dynamics matrix. Show that the output is approximately given by

$$y(t) \approx |G(i\omega)| \sin\big(\omega t + \arg G(i\omega)\big) + \frac{1}{\omega} h(t),$$

where $G(s)$ is the frequency response of the system and $h(t)$ its impulse response.

2.8 Consider the system

$$\frac{dx}{dt} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u, \qquad y = \begin{bmatrix} 1 & 0 \end{bmatrix} x,$$

which is stable but not asymptotically stable. Show that if the system is driven by the bounded input $u = \cos t$ then the output is unbounded.

2.9 Consider a linear system $\dot{x} = Ax$ with the matrix A given by

$$A = \begin{bmatrix} \lambda_1 & 1 \\ 0 & \lambda_2 \end{bmatrix}$$

where $\lambda_1, \lambda_2 \in \mathbb{R}$.

(a) Find the stable, unstable, and center subspaces E^s, E^u, and E^c for $\lambda_1 > 0$ and $\lambda_2 < 0$.

(b) Qualitatively sketch the phase portrait of the system:

i. For $\lambda_1, \lambda_2 > 0$

ii. For $\lambda_1, \lambda_2 < 0$

iii. For $\lambda_1 > 0$ and $\lambda_2 < 0$

(c) Compute the matrix exponential, e^{At} for the system for all $\lambda_1, \lambda_2 \in \mathbb{R}$.

(d) From part (a), verify that $\mathbb{R}^2 = E^s \oplus E^u \oplus E^c$, where $\oplus$ represents the direct-sum of the vector spaces. Also verify that these subspaces are invariant under e^{At}.

(e) Give an example of a *non-hyperbolic* (Definition 2.2 FBS2s) linear system ($\dot{x} = Ax + Bu$, $y = Cx$). For all bounded inputs to your system, is the output bounded? Prove or give a counter example.

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