Feedback Systems

An Introduction for Scientists and Engineers

SECOND EDITION

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Version v3.1.5 (2020-07-24)

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Chapter 9

Transfer Functions

The typical regulator system can frequently be described, in essentials, by differential equations of no more than perhaps the second, third, or fourth order. . . . In contrast, the order of the set of differential equations describing the typical negative feedback amplifier used in telephony is likely to be very much greater. As a matter of idle curiosity, I once counted to find out what the order of the set of equations in an amplifier I had just designed would have been, if I had worked with the differential equations directly. It turned out to be 55.

Hendrik Bode, 1960 [Bod60].

This chapter introduces the concept of the *transfer function*, which is a compact description of the input/output relation for a linear time-invariant system. We show how to obtain transfer functions analytically and experimentally. Combining transfer functions with block diagrams gives a powerful algebraic method to analyze linear systems with many blocks. The transfer function allows new interpretations of system dynamics. We also introduce the Bode plot, a powerful graphical representation of the transfer function that was introduced by Bode to analyze and design feedback amplifiers.

9.1 Frequency Domain Modeling

Figure 9.1 is a block diagram for a typical control system, consisting of a process to be controlled and a controller that combines feedback and feedforward. We saw in the previous two chapters how to analyze and design such systems using state space descriptions of the blocks. As mentioned in Chapter 3, an alternative approach is to focus on the input/output characteristics of the system. Since it is the inputs and outputs that are used to connect the systems, one could expect that this point of view would allow an understanding of the overall behavior of the system. Transfer functions are the main tool in implementing this approach for linear systems.

The basic idea of the transfer function comes from looking at the frequency response of a system. Suppose that we have an input signal that is periodic. Then

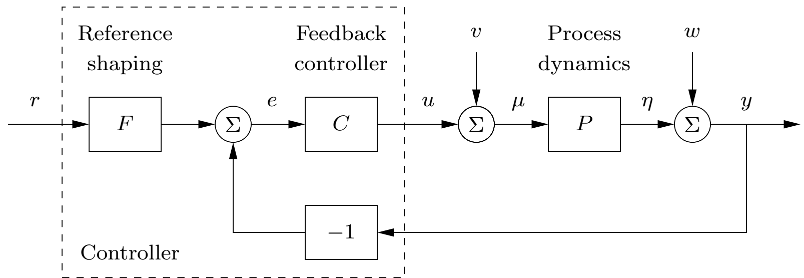


Figure 9.1: A block diagram for a feedback control system. The reference signal r is fed through a reference shaping block, which generates a signal that is compared with the output y to form the error e. The control signal u is generated by the controller, which has the error as the input. The load disturbance v and the measurement noise w are external signals.

we can decompose this signal into the sum of a set of sines and cosines,

$$u(t) = \sum_{k=0}^{\infty} a_k \sin(k\omega_{\rm f} t) + b_k \cos(k\omega_{\rm f} t),$$

where $\omega_{\rm f}$ is the fundamental frequency of the periodic input. As we saw in Section 6.3, the input $u(t)$ generates corresponding sine and cosine outputs (in steady state), with possibly shifted magnitude and phase. The gain and phase at each frequency are determined by the frequency response given in equation (6.24):

$$G(i\omega) = C(i\omega I - A)^{-1}B + D, \tag{9.1}$$

where we set $\omega = k\omega_{\rm f}$ for each $k = 1, \dots, \infty$. We can thus use the steady-state frequency response $G(i\omega)$ and superposition to compute the steady-state response any periodic signal.

The transfer function generalizes this notion to allow a broader class of input signals besides periodic ones. As we shall see in the next section, the transfer function represents the response of the system to an *exponential input*, $u = e^{st}$. It turns out that the form of the transfer function is precisely the same as that of equation (9.1). This should not be surprising since we derived equation (9.1) by writing sinusoids as sums of complex exponentials. The transfer function can also be introduced as the ratio of the Laplace transforms of the output and the input when the initial state is zero, although one does not have to understand the details of Laplace transforms in order to make use of transfer functions.

Modeling a system through its response to sinusoidal and exponential signals is known as *frequency domain modeling*. This terminology stems from the fact that we represent the dynamics of the system in terms of the generalized frequency s rather than the time domain variable t. The transfer function provides a complete representation of a linear system in the frequency domain.

The power of transfer functions is that they provide a particularly convenient representation in manipulating and analyzing complex linear feedback systems. As we shall see, there are graphical representations of transfer functions (Bode and Nyquist plots) that capture interesting properties of the underlying dynamics.

Transfer functions also make it possible to express the changes in a system because of modeling error, which is essential when considering sensitivity to process variations of the sort discussed in Chapter 13. More specifically, using transfer functions it is possible to analyze what happens when dynamical models are approximated by static models or when high-order models are approximated by low-order models. One consequence is that we can introduce concepts that express the degree of stability of a system.

While many of the concepts for state space modeling and analysis apply directly to nonlinear systems, frequency domain analysis applies primarily to linear systems. The notions of gain and phase can, however, be generalized to nonlinear systems and, in particular, propagation of sinusoidal signals through a nonlinear system can approximately be captured by an analog of the frequency response called the describing function. These extensions of frequency response will be discussed in Section 10.5.

9.2 Determining the Transfer Function

As we have seen in previous chapters, the input/output dynamics of a linear system have two components: the initial condition response and the forced response, which depends on the system input. The forced response can be characterized by the transfer function. In this section we will compute transfer functions for general linear time-invariant systems. Transfer functions will also be determined for systems with time delays and systems described by partial differential equations, for which the transfer functions obtained are then transcendental functions of a complex variable.

Transmission of Exponential Signals

To formally compute the transfer function of a system, we will make use of a special type of signal, called an *exponential signal,* of the form e^{st}, where $s = \sigma + i\omega$ is a complex number. Exponential signals play an important role in linear systems. They appear in the solution of differential equations and in the impulse response of linear systems, and many signals can be represented as exponentials or sums of exponentials. For example, a constant signal is simply $e^{\alpha t}$ with $\alpha = 0$. Using Euler's formula, damped sine and cosine signals can be represented by

$$e^{(\sigma+i\omega)t} = e^{\sigma t}e^{i\omega t} = e^{\sigma t}(\cos\omega t + i\sin\omega t),$$

where $\sigma < 0$ determines the decay rate. Figure 9.2 gives examples of signals that can be represented by complex exponentials; many other signals can be represented by linear combinations of these signals.

As in the case of the sinusoidal signals we considered in Section 6.3, we will allow complex-valued signals in the derivation that follows, although in practice we always add together combinations of signals that result in real-valued functions. Ⓕ

To find the transfer function for the state space system

$$\frac{dx}{dt} = Ax + Bu, \qquad y = Cx + Du, \tag{9.2}$$

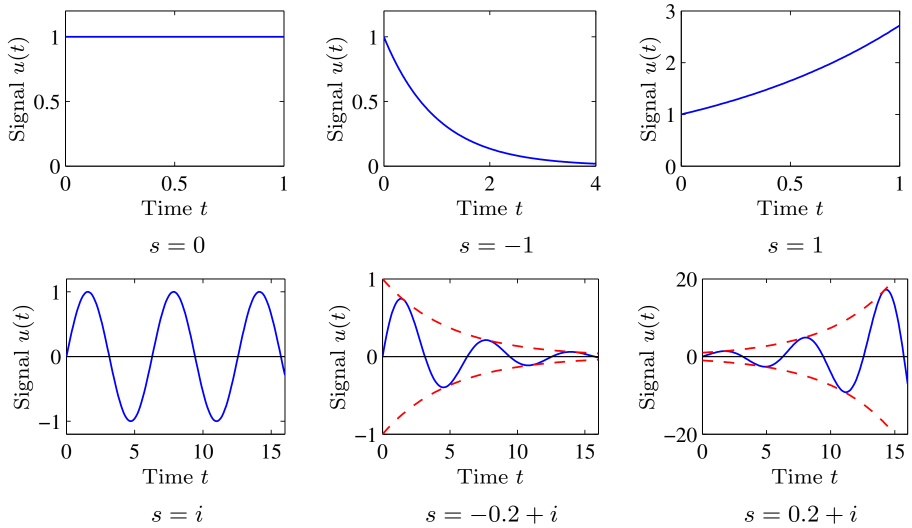


Figure 9.2: Examples of exponential signals. The top row corresponds to exponential signals with a real exponent, and the bottom row corresponds to those with complex exponents. The dashed line in the last two cases denotes the bounding envelope for the oscillatory signals. In each case, if the real part of the exponent is negative then the signal decays, while if the real part is positive then it grows.

we let the input be the exponential signal $u(t) = e^{st}$ and assume that $s \notin \lambda(A)$. The state is then given by

$$x(t) = e^{At}x(0) + \int_0^t e^{A(t-\tau)}Be^{s\tau}\,d\tau = e^{At}x(0) + e^{At}(sI-A)^{-1}\Big(e^{(sI-A)t} - I\Big)B.$$

The output y of equation (9.2) then becomes

$$\begin{aligned} y(t) &= Cx(t) + Du(t) \\ &= \underbrace{Ce^{At}x(0)}_{\substack{\text{initial condition}\\ \text{response}}} + \underbrace{\Big(C(sI-A)^{-1}B + D\Big)e^{st} - Ce^{At}(sI-A)^{-1}B}_{\text{input response}} \\ &= \underbrace{Ce^{At}\Big(x(0) - (sI-A)^{-1}B\Big)}_{\text{transient response}} + \underbrace{\Big(C(sI-A)^{-1}B + D\Big)e^{st}}_{\text{pure exponential response } y_{\text{p}}}, \end{aligned} \tag{9.3}$$

and the *transfer function* from u to y of the system (9.2) is the coefficient of the term e^{st}, hence

$$G(s) = C(sI-A)^{-1}B + D. \tag{9.4}$$

Compare this with the definition of frequency response given by equations (6.23) and (6.24).

An important point in the derivation of the transfer function is the fact that we have restricted s so that $s \neq \lambda_j(A)$, the eigenvalues of A. At those values of s, we see that the response (9.3) of the system is singular (since $sI - A$ then is not invertible). The transfer function can, however, be extended to all values of s by analytic continuation.

To give some insight we will now discuss the structure of equation (9.3). We first notice that the output $y(t)$ can be separated into two terms in two different ways, as is indicated by braces in the equation.

The response of the system to initial conditions is $Ce^{At}x(0)$. Recall that e^{At} can be written in terms of the eigenvalues of A (using the Jordan form in the case of repeated eigenvalues), and hence the transient response is a linear combination of terms of the form $p_j(t)e^{\lambda_j t}$, where λ_j are eigenvalues of A and $p_j(t)$ is a polynomial whose degree is less than the multiplicity of the eigenvalue (Exercise 9.1).

The response to the input $u(t) = e^{st}$ contains a mixture of terms $p_j(t)e^{\lambda_j t}$ and the exponential function

$$y_{\mathrm{p}}(t) = \big(C(sI - A)^{-1}B + D\big)e^{st} = G(s)e^{st}, \tag{9.5}$$

which is a *particular solution* to the differential equation (9.2). We call equation (9.5) the *pure exponential solution* because it has only one exponential e^{st}. It follows from equation (9.3) that the output $y(t)$ is equal to the pure exponential solution $y_{\mathrm{p}}(t)$ if the initial condition is chosen as

$$x(0) = (sI - A)^{-1}B. \tag{9.6}$$

If the system (9.2) is asymptotically stable, then $e^{At} \to 0$ as $t \to \infty$. If in addition the input $u(t)$ is a constant $u(t) = e^{0\cdot t}$ or a sinusoid $u(t) = e^{i\omega t}$ then the response converges to a constant or sinusoidal *steady-state solution* (as shown in equation (6.23)).

To simplify manipulation of the equations describing linear time-invariant systems, we introduce $\mathcal{E}$ as the class of time functions that can be created from combinations of signals of the form $X(s)e^{st}$, where the parameter s is a complex variable and $X(s)$ is a complex function (vector valued if needed). It follows from equations (9.3) and (9.4) that if a system with transfer function $G(s)$ has the input $u \in \mathcal{E}$ then there is a particular solution $y \in \mathcal{E}$ that satisfies the dynamics of the system. This solution is the actual response of the system if the initial condition is chosen as equation (9.6). Since the transfer function of a system is given by the pure exponential response, we can derive transfer functions using exponential signals, and we will use the notation

$$y = G_{yu}\, u, \tag{9.7}$$

where G_{yu} is the transfer function for the linear input/output system taking u to y. Mathematically, it is important to remember that this notation assumes the use of combinations of exponential signals. We will also often drop the subscripts on G and just write $y = Gu$ when the meaning is clear from context.

Example 9.1 Damped oscillator

Consider the response of a damped linear oscillator, whose state space dynamics were studied in Section 7.3:

$$\frac{dx}{dt} = \begin{pmatrix} 0 & \omega_0 \\ -\omega_0 & -2\zeta\omega_0 \end{pmatrix} x + \begin{pmatrix} 0 \\ k\omega_0 \end{pmatrix} u, \qquad y = \begin{pmatrix} 1 & 0 \end{pmatrix} x. \tag{9.8}$$

This system is asymptotically stable if $\zeta > 0$, and so we can look at the steady-state response to an input $u = e^{st}$:

$$\begin{aligned} G_{yu}(s) = C(sI - A)^{-1}B &= \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} s & -\omega_0 \\ \omega_0 & s + 2\zeta\omega_0 \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ k\omega_0 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \end{pmatrix} \left(\frac{1}{s^2 + 2\zeta\omega_0 s + \omega_0^2} \begin{pmatrix} s + 2\zeta\omega_0 & -\omega_0 \\ \omega_0 & s \end{pmatrix} \right) \begin{pmatrix} 0 \\ k\omega_0 \end{pmatrix} \\ &= \frac{k\omega_0^2}{s^2 + 2\zeta\omega_0 s + \omega_0^2}. \end{aligned} \tag{9.9}$$

The steady-state response to a step input is obtained by setting $s = 0$, which gives

$$u = 1 \qquad \Longrightarrow \qquad y = G_{yu}(0)u = k.$$

If we wish to compute the steady-state response to a sinusoid, we write

$$u = \sin\omega t = \frac{1}{2}\left(ie^{-i\omega t} - ie^{i\omega t}\right) \qquad \Longrightarrow \qquad y = \frac{1}{2}\left(iG_{yu}(-i\omega)e^{-i\omega t} - iG_{yu}(i\omega)e^{i\omega t}\right).$$

We can now write $G(i\omega)$ in terms of its magnitude and phase,

$$G(i\omega) = \frac{k\omega_0^2}{-\omega^2 + (2\zeta\omega_0\omega)i + \omega_0^2} = Me^{i\theta},$$

where the magnitude (or gain) M and phase θ are given by

$$M = \frac{k\omega_0^2}{\sqrt{(\omega_0^2 - \omega^2)^2 + (2\zeta\omega_0\omega)^2}}, \qquad \frac{\sin\theta}{\cos\theta} = \frac{-2\zeta\omega_0\omega}{\omega_0^2 - \omega^2}.$$

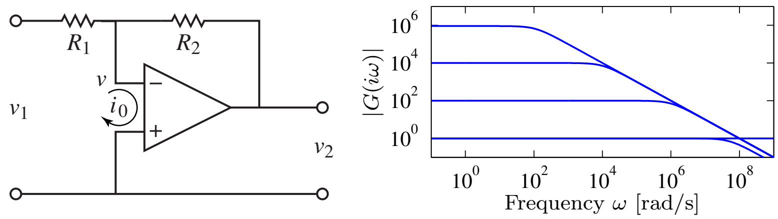
We can also make use of the fact that $G(-i\omega)$ is given by its complex conjugate $G^*(i\omega)$, and it follows that $G(-i\omega) = Me^{-i\theta}$. Substituting these expressions into our output equation, we obtain

$$\begin{aligned} y &= \frac{1}{2}\left(i(Me^{-i\theta})e^{-i\omega t} - i(Me^{i\theta})e^{i\omega t}\right) \\ &= M \cdot \frac{1}{2}\left(ie^{-i(\omega t+\theta)} - ie^{i(\omega t+\theta)}\right) = M\sin(\omega t + \theta). \end{aligned}$$

The responses to other signals can be computed by writing the input as an appropriate combination of exponential responses and using linearity. ∇

Example 9.2 Operational amplifier circuit
To further illustrate the use of exponential signals, we consider the operational amplifier circuit described in Section 4.3 and reproduced in Figure 9.3a. The model in Section 4.3 is a simplification because the linear behavior of the amplifier is modeled as a constant gain. In reality there are significant dynamics in the amplifier, and the static model $v_{\text{out}} = -kv$ (equation (4.11)) should therefore be replaced by a dynamical model $v_{\text{out}} = -Gv$. A simple transfer function is

$$G(s) = \frac{ak}{s + a}. \tag{9.10}$$



(a) Circuit diagram (b) Frequency response

Figure 9.3: Stable amplifier based on negative feedback around an operational amplifier. The circuit diagram on the left shows a typical amplifier with low-frequency gain R_2/R_1. If we model the dynamic response of the op amp as $G(s) = ak/(s+a)$, then the gain falls off at frequency $\omega = aR_1k/R_2$, as shown in the gain curves on the right. The frequency response is computed for $k = 10^7$, $a = 10$ rad/s, $R_2 = 10^6$ Ω, and $R_1 = 1$, 10^2, 10^4, and 10^6 Ω.

These dynamics correspond to a first-order system with time constant $1/a$. The parameter k is called the *open loop gain*, and the product ak is called the *gain-bandwidth product*; typical values for these parameters are $k = 10^7$ and $ak = 10^7$–10^9 rad/s.

If the input v_1 is an exponential signal e^{st}, then there are solutions where all signals in the circuit are exponentials, $v, v_1, v_2 \in \mathcal{E}$, since all of the elements of the circuit are modeled as being linear. The equations describing the system can then be manipulated algebraically.

Assuming that the current into the amplifier is zero, as is done in Section 4.3, the currents through the resistors R_1 and R_2 are the same, hence

$$\frac{v_1 - v}{R_1} = \frac{v - v_2}{R_2}, \quad \text{or} \quad (R_1 + R_2)v = R_2 v_1 + R_1 v_2.$$

Combining the above equation with the open loop dynamics of the operational amplifier (9.10), which can be written as $v_2 = -Gv$ in the simplified notation (9.7), gives the following model for the closed loop system:

$$(R_1 + R_2)v = R_2 v_1 + R_1 v_2, \qquad v_2 = -Gv, \qquad v, v_1, v_2 \in \mathcal{E}. \tag{9.11}$$

Eliminating v between these equations yields

$$v_2 = \frac{-R_2 G}{R_1 + R_2 + R_1 G} v_1 = \frac{-R_2 ak}{R_1 ak + (R_1 + R_2)(s + a)} v_1,$$

and the transfer function of the closed loop system is

$$G_{v_2 v_1} = \frac{-R_2 ak}{R_1 ak + (R_1 + R_2)(s + a)}. \tag{9.12}$$

The low-frequency gain is obtained by setting $s = 0$, hence

$$G_{v_2 v_1}(0) = \frac{-kR_2}{(k+1)R_1 + R_2} \approx -\frac{R_2}{R_1},$$

which is the result given by equation (4.12) in Section 4.3. The bandwidth of the amplifier circuit is

$$\omega_{\mathrm{b}} = a\frac{R_1(k+1)+R_2}{R_1+R_2} \approx a\frac{R_1 k}{R_2} \quad \text{for} \quad k \gg 1,$$

where the approximation holds for $R_2/R_1 \gg 1$. The gain of the closed loop system drops off at high frequencies as $R_2ak/(\omega(R_1+R_2))$. The frequency response of the transfer function is shown in Figure 9.3b for $k = 10^7$, $a = 10$ rad/s, $R_2 = 10^6\ \Omega$, and $R_1 = 1$, 10^2, 10^4, and $10^6\ \Omega$.

Note that in solving this example, we bypassed explicitly writing the signals as $v = V(s)e^{st}$ and instead worked directly with v, assuming it was an exponential. This shortcut is handy in solving problems of this sort and when manipulating block diagrams. A comparison with Section 4.3, where we make the same calculation when $G(s)$ is a constant, shows analysis of systems using transfer functions is as easy as using static systems. The calculations are the same if the resistances R_1 and R_2 are replaced by impedances, as discussed further in Example 9.3. ∇

Transfer Functions for Linear Differential Equations

Consider a linear system described by the controlled differential equation

$$\frac{d^n y}{dt^n} + a_1\frac{d^{n-1}y}{dt^{n-1}} + \cdots + a_n y = b_0\frac{d^m u}{dt^m} + b_1\frac{d^{m-1}u}{dt^{m-1}} + \cdots + b_m u, \tag{9.13}$$

where u is the input and y is the output. Notice that here we have generalized our system description from Section 3.2 to allow both the input and its derivatives to appear. This type of description arises in many applications, as described briefly in Chapter 2 and Section 3.2; bicycle dynamics and AFM modeling are two specific examples.

To determine the transfer function of the system (9.13), let the input be $u(t) = e^{st}$. Since the system is linear, there is an output of the system that is also an exponential function $y(t) = y_0e^{st}$. Inserting the signals into equation (9.13), we find

$$(s^n + a_1s^{n-1} + \cdots + a_n)y_0e^{st} = (b_0s^m + b_1s^{m-1}\cdots + b_m)e^{st},$$

and the response of the system can be completely described by two polynomials

$$a(s) = s^n + a_1s^{n-1} + \cdots + a_n, \qquad b(s) = b_0s^m + b_1s^{m-1} + \cdots + b_m. \tag{9.14}$$

The polynomial $a(s)$ is the characteristic polynomial of the ordinary differential equation. If $a(s) \neq 0$, it follows that

$$y(t) = y_0e^{st} = \frac{b(s)}{a(s)}e^{st}. \tag{9.15}$$

The transfer function of the system (9.13) is thus the rational function

$$G(s) = \frac{b(s)}{a(s)} = \frac{b_0s^m + b_1s^{m-1} + \cdots + b_m}{s^n + a_1s^{n-1} + \cdots + a_n}, \tag{9.16}$$

where the polynomials $a(s)$ and $b(s)$ are given by equation (9.14). Notice that the transfer function for the system (9.13) can be obtained by inspection since the coefficients of $a(s)$ and $b(s)$ are precisely the coefficients of the derivatives of u and y. The *poles* and the *zeros* of the transfer function are the roots of the polynomials $a(s)$ and $b(s)$. The properties of the system are determined by the poles and zeros of the transfer function, as we shall see in the examples that follow and shall explore in more detail in Section 9.5.

Example 9.3 Electrical circuit elements

Modeling of electrical circuits is a common use of transfer functions. Consider, for example, a resistor modeled by Ohm's law $V = IR$, where V is the voltage across the resistor, I is the current through the resistor, and R is the resistance value. If we consider current to be the input and voltage to be the output, the resistor has the transfer function $Z(s) = R$, which is also called the *generalized impedance* of the circuit element.

Next we consider an inductor whose input/output characteristic is given by

$$L\frac{dI}{dt} = V.$$

Letting the current be $I(t) = e^{st}$, we find that the voltage is $V(t) = Lse^{st}$ and the transfer function of an inductor is thus $Z(s) = Ls$. A capacitor is characterized by

$$C\frac{dV}{dt} = I,$$

and a similar analysis gives a transfer function from current to voltage of $Z(s) = 1/(Cs)$. Using transfer functions, complex electrical circuits can be analyzed algebraically by using the generalized impedance $Z(s)$ just as one would use the resistance value in a resistor network. ∇

Example 9.4 Vibration damper

Damping vibrations is a common engineering problem. A schematic diagram of a vibration damper is shown in Figure 9.4. To analyze the system we use Newton's equations for the two masses:

$$m_1\ddot{q}_1 + c_1\dot{q}_1 + k_1q_1 + k_2(q_1 - q_2) = F, \qquad m_2\ddot{q}_2 + k_2(q_2 - q_1) = 0.$$

To determine the transfer function from the force F to the position q_1 of the mass m_1 we first find particular exponential solutions:

$$(m_1s^2 + c_1s + k_1)q_1 + k_2(q_1 - q_2) = F, \qquad m_2s^2q_2 + k_2(q_2 - q_1) = 0. \tag{9.17}$$

We solve q_2 from the second expression,

$$q_2 = \frac{k_2}{m_2s^2 + k_2}q_1,$$

and insert this into the first expression to obtain

$$(m_1s^2 + c_1s + k_1)q_1 + k_2\Big(1 - \frac{k_2}{m_2s^2 + k_2}\Big)q_1 = F,$$

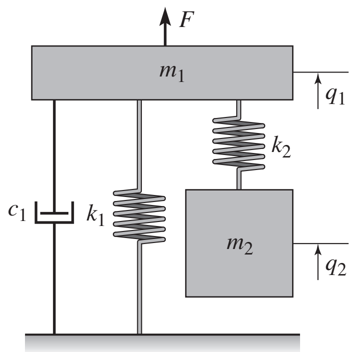


Figure 9.4: A vibration damper. Vibrations of the mass m_1 can be damped by providing it with an auxiliary mass m_2, attached to m_1 by a spring with stiffness k_2. The parameters m_2 and k_2 are chosen so that the frequency $\sqrt{k_2/m_2}$ matches the frequency of the vibration.

and hence

$$\big((m_1s^2 + c_1s + k_1 + k_2)(m_2s^2 + k_2) - k_2^2\big)\, q_1 = (m_2s^2 + k_2)F.$$

Expanding the expression gives the transfer function

$$G_{q_1F}(s) = \frac{m_2s^2 + k_2}{m_1m_2s^4 + m_2c_1s^3 + (m_1k_2 + m_2(k_1 + k_2))s^2 + k_2c_1s + k_1k_2}$$

from the disturbance force F to the position q_1 of the mass m_1. The transfer function has a zero at $s = \pm i\sqrt{k_2/m_2}$, which means that transmission of sinusoidal signals with this frequency are blocked (this blocking property will be discussed in Section 9.5). ∇

As the examples above illustrate, transfer functions provide a simple representation for linear input/output systems. Transfer functions for some common linear time-invariant systems are given in Table 9.1. Transfer functions of a form similar to equation (9.13) can also be constructed for systems with many inputs and many outputs.

Time Delays and Partial Differential Equations

Although we have focused thus far on ordinary differential equations, transfer functions can also be used for other types of linear systems. We illustrate this using time delays and systems described by partial differential equations.

Example 9.5 Time delay
Time delays appear in many systems: typical examples are delays in nerve propagation, communication systems, and mass transport. A system with a time delay has the input/output relation

$$y(t) = u(t - \tau). \tag{9.18}$$

Table 9.1: Transfer functions for some common linear time-invariant systems.

Type	System	Transfer Function
Integrator	$\dot{y} = u$	$\frac{1}{s}$
Differentiator	$y = \dot{u}$	s
First-order system	$\dot{y} + ay = u$	$\frac{1}{s+a}$
Double integrator	$\ddot{y} = u$	$\frac{1}{s^2}$
Damped oscillator	$\ddot{y} + 2\zeta\omega_0\dot{y} + \omega_0^2 y = u$	$\frac{1}{s^2 + 2\zeta\omega_0 s + \omega_0^2}$
State space system	$\dot{x} = Ax + Bu,\ y = Cx + Du$	$C(sI - A)^{-1}B + D$
PID controller	$y = k_\text{p}u + k_\text{d}\dot{u} + k_\text{i}\int u$	$k_\text{p} + k_\text{d}s + \frac{k_\text{i}}{s}$
Time delay	$y(t) = u(t - \tau)$	$e^{-\tau s}$

To obtain the corresponding transfer function we let the input be $u(t) = e^{st}$, and the output is then

$$y(t) = u(t-\tau) = e^{s(t-\tau)} = e^{-s\tau}e^{st} = e^{-s\tau}u(t).$$

We find that the transfer function of a time delay is thus $G(s) = e^{-s\tau}$, which is not a rational function. ∇

Example 9.6 Heat propagation

Consider the problem of one-dimensional heat propagation in a semi-infinite metal rod. Assume that the input is the temperature at one end and that the output is the temperature at a point along the rod. Let $\theta(x,t)$ be the temperature at position x and time t. With a proper choice of length scales and units, heat propagation is described by the partial differential equation

$$\frac{\partial\theta}{\partial t} = \frac{\partial^2\theta}{\partial^2 x}, \qquad y(t) = \theta(1,t), \tag{9.19}$$

and the point of interest can be assumed to have $x = 1$. The boundary condition for the partial differential equation is

$$\theta(0,t) = u(t).$$

To determine the transfer function we choose the input as $u(t) = e^{st}$. Assume that there is a solution to the partial differential equation of the form $\theta(x,t) = \psi(x)e^{st}$ and insert this into equation (9.19) to obtain

$$s\psi(x) = \frac{d^2\psi}{dx^2},$$

with boundary condition $\psi(0) = 1$. This ordinary differential equation (with independent variable x) has the solution

$$\psi(x) = Ae^{x\sqrt{s}} + Be^{-x\sqrt{s}}.$$

Since the temperature of the rod is bounded we have $A = 0$, the boundary condition gives $B = 1$, and the solution is then

$$y(t) = \theta(1,t) = \psi(1)e^{st} = e^{-\sqrt{s}}e^{st} = e^{-\sqrt{s}}u(t).$$

The system thus has the transfer function $G(s) = e^{-\sqrt{s}}$. As in the case of a time delay, the transfer function is not a rational function. ∇

State Space Realizations of Transfer Functions

We have seen in equation (9.4) how to compute the transfer function for a given state space control system. The inverse problem, computing a state space control system for a given transfer function, is known as the *realization problem*. Given a transfer function $G(s)$, we say that a state space system with matrices A, B, C, and D is a (state space) *realization* of $G(s)$ if $G(s) = C(sI - A)^{-1}B + D$. We explore here some of the properties of realizations of transfer functions, starting with the question of uniqueness.

As we saw in Section 6.3, it is possible to choose a different set of coordinates for the state space of a linear system and still preserve the input/output response. In other words, the matrices A, B, C, and D in the state space equations (9.2) depend on the choice of coordinate system used for the states, but since the transfer function relates input to outputs, it should be invariant to coordinate changes in the state space. Repeating the analysis in Chapter 6, consider a model (9.2) and introduce new coordinates z by the transformation $z = Tx$, where T is a nonsingular matrix. The system is then described by

$$\begin{aligned}\frac{dz}{dt} &= T(Ax + Bu) = TAT^{-1}z + TBu =: \tilde{A}z + \tilde{B}u,\\ y &= Cx + Du = CT^{-1}z + Du =: \tilde{C}z + Du.\end{aligned}$$

This system has the same form as equation (9.2), but the matrices A, B, and C are different:

$$\tilde{A} = TAT^{-1}, \qquad \tilde{B} = TB, \qquad \tilde{C} = CT^{-1}. \tag{9.20}$$

Computing the transfer function of the transformed model, we get

$$\begin{aligned}\tilde{G}(s) &= \tilde{C}(sI - \tilde{A})^{-1}\tilde{B} + D = CT^{-1}(sI - TAT^{-1})^{-1}TB + D\\ &= C\big(T^{-1}(sI - TAT^{-1})T\big)^{-1}B + D = C(sI - A)^{-1}B + D = G(s),\end{aligned}$$

which is identical to the transfer function (9.4) computed from the system description (9.2). The transfer function is thus invariant to changes of the coordinates in the state space.

One consequence of this coordinate invariance is that it is not possible for there to be a *unique* state space realization for a given transfer function. Given any one realization, we can compute another realization by simply changing coordinates using any invertible matrix T. Note, however, that the dimension of the state space realization is not changed by this transformation. It therefore makes sense to talk about a *minimal realization*, in which the number of states is as small as possible. For a transfer function $G(s) = b(s)/a(s)$ with denominator $a(s)$ of degree n, it can be shown that there is always a realization with n states, given by a state space system in reachable canonical form (7.6). In general, a minimal realization will always have at most n states. However, the degree may be lower if there are pole/zero cancellations, as illustrated by the following example.

Example 9.7 Cancellation of poles and zeros
Consider the system

$$\frac{dx}{dt} = \begin{pmatrix} -3 & 1 \\ -2 & 0 \end{pmatrix} x + \begin{pmatrix} 1 \\ 1 \end{pmatrix} u, \qquad y = \begin{pmatrix} 1 & 0 \end{pmatrix} x.$$

Equation (9.4) gives the following transfer function

$$\begin{aligned} G(s) &= \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} s+3 & -1 \\ 2 & s \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{1}{s^2+3s+2} \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} s & 1 \\ -2 & s+3 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \\ &= \frac{s+1}{s^2+3s+2} = \frac{s+1}{(s+1)(s+2)} = \frac{1}{s+2}. \end{aligned}$$

Even though the original state space system was of second order, the transfer function is a first-order rational function. The reason is that the factor $s + 1$ has been canceled when computing the transfer function. Cancellation of poles and zeros is related to lack of reachability and observability. In this particular case the reachability matrix

$$W_\mathrm{r} = \begin{pmatrix} B & AB \end{pmatrix} = \begin{pmatrix} 1 & -2 \\ 1 & -2 \end{pmatrix}$$

has rank 1 and the system is not reachable. Notice that it was shown in Section 8.3 that the transfer function is given by the reachable and observable subsystem Σ_{ro} in the Kalman decomposition of a linear system, which in this case is of first order. ∇

The general approach to understand realizations (and minimal realizations) is to make use of the Kalman decomposition in Section 8.3. We see from the structure of equation (8.20) that the input/output response of a linear control system is determined solely by the reachable and observable subsystem Σ_{ro}. When a system lacks reachability and observability, this shows up as cancellation of poles and zeros in the transfer function computed from the full system matrices.

Cancellation of poles and zeros was controversial for a long time, which was manifested in rules for manipulating transfer functions: do not cancel factors with roots in the right half-plane. Special algebraic methods were also developed to do block diagram algebra. Kalman's decomposition, which clarifies that the transfer function only represents part of the dynamics, gives clear insight into what is happening. These issues are discussed in more detail in Section 9.5.

The results of this section can also be extended to the case of multi-input, multi-output (MIMO) systems. The transfer function $G(s)$ for a single-input, single-output given by equation (9.4) is a function of complex variables, $G : \mathbb{C} \to \mathbb{C}$.

Table 9.2: Laplace transforms for some common signals.

Signal $u(t)$	Laplace transform $U(s)$	Signal $u(t)$	Laplace transform $U(s)$
$S(t)$ [unit step]	$\dfrac{1}{s}$	$\delta(t)$ [impulse]	1
$\sin(at)$	$\dfrac{a}{s^2+a^2}$	$\cos(at)$	$\dfrac{s}{s^2+a^2}$
$e^{-\alpha t}\sin(at)$	$\dfrac{a}{(s+\alpha)^2+a^2}$	$e^{-\alpha t}\cos(at)$	$\dfrac{s+\alpha}{(s+\alpha)^2+a^2}$

For systems with p inputs and q outputs the transfer function is matrix-valued, $G : \mathbb{C} \to \mathbb{C}^{q\times p}$. The techniques described above can be generalized to this case, but the notion of a (minimal) realization becomes substantially more complicated.

9.3 Laplace Transforms

The traditional way to derive the transfer function for a linear, time-invariant, input/output system is to make use of Laplace transforms. The Laplace transform method was particularly important before the advent of computers, since it provided a practical way to compute the response of a system to a given input. Today, we compute the response of a linear (or nonlinear) system to complex inputs using numerical simulation, and the Laplace transform is no longer needed for this purpose. It is however, still useful to gain insight into the response of linear systems.

In this section, we provide a brief introduction to the use of Laplace transforms and their connections with transfer functions. Only a few elementary properties of Laplace transforms are needed for basic control applications; students who are not familiar with them can safely skip this section. A good reference for the mathematical material in this section is the classic book by Widder [Wid41] or the more modern treatments available in standard textbooks on signals and systems [LV11, OWN96].

Consider a function $f(t)$, $f : \mathbb{R}^+ \to \mathbb{R}$, that is integrable and grows no faster than $e^{s_0 t}$ for some finite $s_0 \in \mathbb{R}$ and large t. The Laplace transform maps f to a function $F = \mathcal{L}f : \mathbb{C} \to \mathbb{C}$ of a complex variable. It is defined by

$$F(s) = \int_0^\infty e^{-st} f(t)\,dt, \quad \operatorname{Re} s > s_0. \tag{9.21}$$

Using this formula, it is possible to compute the Laplace transform of some common functions; see Table 9.2.

The Laplace transform has some properties that makes it well suited to deal with linear systems. First we observe that the transform itself is linear because

$$\begin{aligned} \mathcal{L}(af+bg) &= \int_0^\infty e^{-st}(af(t)+bg(t))\,dt \\ &= a\int_0^\infty e^{-st} f(t)\,dt + b\int_0^\infty e^{-st} g(t)\,dt = a\mathcal{L}f + b\mathcal{L}g. \end{aligned} \tag{9.22}$$

Using linearity we can compute the Laplace transform of combinations of simple inputs, such as those that make up the set of exponential signals $\mathcal{E}$ introduced earlier.

Next we will calculate the Laplace transform of the integral of a function. Using integration by parts, we get

$$\begin{aligned}\mathcal{L}\int_0^t f(\tau)\,d\tau &= \int_0^\infty \Big(e^{-st}\int_0^t f(\tau)\,d\tau\Big)\,dt\\ &= -\frac{e^{-st}}{s}\int_0^t f(\tau)\,d\tau\Big|_0^\infty + \int_0^\infty \frac{e^{-s\tau}}{s} f(\tau)\,d\tau = \frac{1}{s}\int_0^\infty e^{-s\tau} f(\tau)\,d\tau,\end{aligned}$$

hence Ⓕ

$$\mathcal{L}\int_0^t f(\tau)\,d\tau = \frac{1}{s}\mathcal{L}f = \frac{1}{s}F(s). \tag{9.23}$$

Integration of a time function thus corresponds to division of the corresponding Laplace transform by s.

Since integration corresponds to division by s, we can expect that differentiation corresponds to multiplication by s. This is not quite true as we will see by calculating the Laplace transform of the derivative of a function. We have

$$\mathcal{L}\frac{df}{dt} = \int_0^\infty e^{-st} f'(t)\,dt = e^{-st} f(t)\Big|_0^\infty + s\int_0^\infty e^{-st} f(t)\,dt = -f(0) + s\mathcal{L}f,$$

where the second equality is obtained using integration by parts. We thus obtain

$$\mathcal{L}\frac{df}{dt} = s\mathcal{L}f - f(0) = sF(s) - f(0). \tag{9.24}$$

Notice the appearance of the initial value $f(0)$ of the function. The formula (9.24) is particularly simple if the initial conditions are zero, because if $f(0) = 0$ it follows that differentiation of a function corresponds to multiplication of the transform by s.

Using Laplace transforms the transfer function for a linear time-invariant system can be defined as the ratio of the transform of the input and the output, when the transforms are computed under the assumption that all initial conditions are zero. We will now illustrate how Laplace transforms can be used to compute transfer functions.

Example 9.8 Transfer function of state space model
Consider the state space system described by equation (9.2). Taking Laplace transforms gives

$$sX(s) - x(0) = AX(s) + BU(s), \qquad Y(s) = CX(s) + DU(s).$$

Elimination of $X(s)$ gives

$$X(s) = (sI - A)^{-1}x(0) + (sI - A)^{-1}BU(s). \tag{9.25}$$

When the initial condition $x(0)$ is zero we have

$$X(s) = (sI - A)^{-1}BU(s), \qquad Y(s) = \Big(C(sI - A)^{-1}B + D\Big)U(s),$$

and the transfer function is given by $G(s) = C(sI - A)^{-1}B + D$ (compare with equation (9.4)). ∇

Example 9.9 Transfer functions and impulse response
Consider a linear time-invariant system with zero initial state. We saw in Section 6.3 that the relation between the input u and the output y is given by the convolution integral

$$y(t) = \int_0^\infty h(t-\tau)u(\tau)\,d\tau,$$

where $h(t)$ is the impulse response for the system (assumed causal). Taking the Laplace transform of this expression and using the fact that $h(t') = 0$ for $t' = t - \tau < 0$ gives

$$\begin{aligned}
Y(s) &= \int_0^\infty e^{-st}y(t)\,dt = \int_0^\infty e^{-st}\int_0^\infty h(t-\tau)u(\tau)\,d\tau\,dt \\
&= \int_0^\infty \int_0^t e^{-s(t-\tau)}e^{-s\tau}h(t-\tau)u(\tau)\,d\tau\,dt \\
&= \int_0^\infty \int_0^\infty e^{-st'}h(t')e^{-s\tau}u(\tau)\,d\tau\,dt' \\
&= \int_0^\infty e^{-st}h(t)\,dt \int_0^\infty e^{-s\tau}u(\tau)\,d\tau = H(s)U(s).
\end{aligned}$$

Thus, the input/output response is given by $Y(s) = H(s)U(s)$, where H, U, and Y are the Laplace transforms of h, u, and y.

The system theoretic interpretation is that the Laplace transform of the output of a linear system is a product of two terms, the Laplace transform of the input $U(s)$ and the Laplace transform of the impulse response of the system $H(s)$. A mathematical interpretation is that the Laplace transform of a convolution is the product of the transforms of the functions that are convolved. The fact that the formula $Y(s) = H(s)U(s)$ is much simpler than a convolution is one reason why Laplace transforms have traditionally been popular in engineering. ∇

A variety of theorems are available using Laplace transforms that are useful in a control systems setting. The *initial value theorem* states that

$$\lim_{t\to 0} f(t) = \lim_{s\to\infty} sF(s).$$

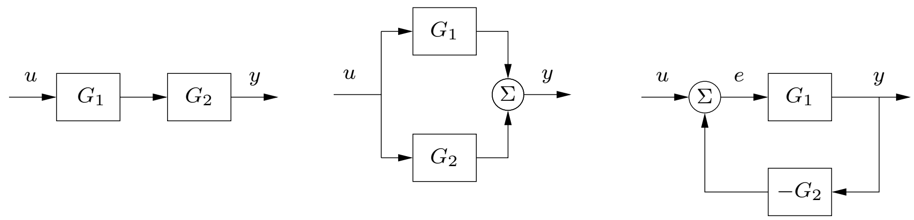
Using this theorem and the fact that a step input has Laplace transform $1/s$, we can compute the initial value of signals in a control system in response to step inputs. For example, if G_{ur} represents that transfer function between the reference r and control input u, then the step response will have the property that

$$u(0) = \lim_{t\to 0} u(t) = \lim_{s\to\infty} sU(s) = \lim_{s\to\infty} s\cdot G_{ur}(s)\cdot\frac{1}{s} = G_{ur}(\infty).$$

Similarly, the *final value theorem* states that

$$\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s),$$

and this can be used to show that for a step input $r(t)$ we have $\lim_{t\to\infty} y(t) = G_{yr}(0)$.



(a) $G_{yu} = G_2 G_1$ (b) $G_{yu} = G_1 + G_2$ (c) $G_{yu} = \dfrac{G_1}{1 + G_1 G_2}$

Figure 9.5: Interconnections of linear systems. Series (a), parallel (b), and feedback (c) connections are shown. The transfer functions for the composite systems can be derived by algebraic manipulations assuming exponential functions for all signals.

9.4 Block Diagrams and Transfer Functions

The combination of block diagrams and transfer functions is a powerful way to represent control systems. Transfer functions relating different signals in the system can be derived by purely algebraic manipulations of the transfer functions of the blocks using *block diagram algebra*. Outputs resulting from several input signals can be derived using *superposition*. To show how this can be done, we will begin with simple combinations of systems. We will assume that all signals are exponential signals $\mathcal{E}$ and we will use the compact notation $y = Gu$ for the output $y \in \mathcal{E}$ of a linear time-invariant system with the input $u \in \mathcal{E}$ and the transfer function G (see equation (9.7) and recall its interpretation).

Consider a system that is a cascade combination of systems with the transfer functions $G_1(s)$ and $G_2(s)$, as shown in Figure 9.5a. Let the input of the system be $u \in \mathcal{E}$. The output of the first block is then $G_1 u \in \mathcal{E}$, which is also the input to the second system. The output of the second system is then

$$y = G_2(G_1 u) = (G_2 G_1)u. \tag{9.26}$$

The transfer function of the series connection is thus $G = G_2 G_1$, i.e., the product of the transfer functions. The order of the individual transfer functions is due to the fact that we place the input signal on the right-hand side of this expression, hence we first multiply by G_1 and then by G_2. Unfortunately, this has the opposite ordering from the diagrams that we use, where we typically have the signal flow from left to right, so one needs to be careful. The ordering is important if either G_1 or G_2 is a vector-valued transfer function, as we shall see in some examples. Ⓗ

Consider next a parallel connection of systems with the transfer functions G_1 and G_2, as shown in Figure 9.5b, and assume that all signals are exponential signals. The outputs of the first and second systems are simply $G_1 u$ and $G_2 u$ and the output of the parallel connection is

$$y = G_1 u + G_2 u = (G_1 + G_2)u.$$

The transfer function for a parallel connection is thus $G = G_1 + G_2$.

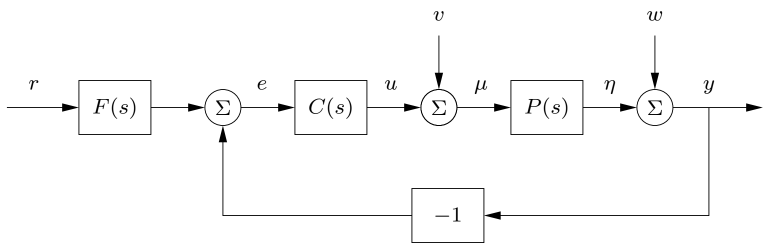


Figure 9.6: Block diagram of a feedback system. The inputs to the system are the reference signal r, the process disturbance v, and the measurement noise w. The remaining signals in the system can all be chosen as possible outputs, and transfer functions can be used to relate the system inputs to the other labeled signals.

Finally, consider a feedback connection of systems with the transfer functions G_1 and G_2, as shown in Figure 9.5c. Writing the relations between the signals for the different blocks and the summation unit, we find

$$y = G_1 e, \qquad e = u - G_2 y. \tag{9.27}$$

Elimination of e gives

$$y = G_1(u - G_2 y) \quad \Longrightarrow \quad (1 + G_1G_2)y = G_1 u \quad \Longrightarrow \quad y = \frac{G_1}{1 + G_1G_2}u.$$

The transfer function of the feedback connection is thus

$$G = \frac{G_1}{1 + G_1G_2}. \tag{9.28}$$

These three basic interconnections can be used as the basis for computing transfer functions for more complicated systems.

Control System Transfer Functions

Consider the system in Figure 9.6, which was given at the beginning of the chapter. The system has three blocks representing a process P, a feedback controller C, and a feedforward controller F. Together, C and F define the *control law* for the system. There are three external signals: the reference (or command) signal r, the load disturbance v, and the measurement noise w. A typical problem is to determine how the error e is related to the signals r, v, and w.

To derive the transfer functions we are interested in, we assume that all signals are exponential signals $\mathcal{E}$ and we write the relations between the signals for each block in the system block diagram. Assume for example that we are interested in the control error e. The summation point and the block $F(s)$ gives

$$e = Fr - y.$$

The signal y is the sum of w and η, where η is the output of the process $P(s)$:

$$y = w + \eta, \qquad \eta = P(v + u), \qquad u = Ce.$$

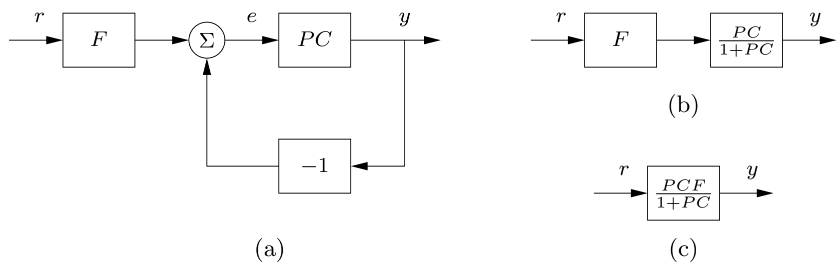


Figure 9.7: Example of block diagram algebra. The results from multiplying the process and controller transfer functions (from Figure 9.6) are shown in (a). Replacing the feedback loop with its transfer function equivalent yields (b), and finally multiplying the two remaining blocks gives the reference to output representation in (c).

Combining these equations gives

$$\begin{aligned} e &= Fr - y = Fr - (w + \eta) = Fr - \big(w + P(v + u)\big) \\ &= Fr - \big(w + P(v + Ce)\big), \end{aligned}$$

and hence

$$e = Fr - w - Pv - PCe.$$

Finally, solving this equation for e gives

$$e = \frac{F}{1+PC}\,r - \frac{1}{1+PC}\,w - \frac{P}{1+PC}\,v = G_{er}r + G_{ew}w + G_{ev}v, \tag{9.29}$$

and the error is thus the sum of three terms, depending on the reference r, the measurement noise w, and the load disturbance v. The functions

$$G_{er} = \frac{F}{1+PC}, \qquad G_{ew} = \frac{-1}{1+PC}, \qquad G_{ev} = \frac{-P}{1+PC} \tag{9.30}$$

are transfer functions from reference r, noise w, and disturbance v to the error e. Equation (9.29) can also be obtained by computing the outputs for each input and using superposition.

We can also derive transfer functions by manipulating the block diagrams directly, as illustrated in Figure 9.7. Suppose we wish to compute the transfer function between the reference r and the output y. We begin by combining the process and controller blocks in Figure 9.6 to obtain the diagram in Figure 9.7a. We can now eliminate the feedback loop using the algebra for a feedback interconnection (Figure 9.7b) and then use the series interconnection rule to obtain

$$G_{yr} = \frac{PCF}{1+PC}. \tag{9.31}$$

Similar manipulations can be used to obtain the other transfer functions (Exercise 9.10).

The above analysis illustrates an effective way to manipulate the equations to obtain the relations between inputs and outputs in a feedback system. The general idea is to start with the variable of interest and to trace variables backwards around the feedback loop. With some practice, equations (9.29) and (9.30) can be written directly by inspection of the block diagram. Notice, for example, that all terms in equation (9.30) have the same denominator and that the numerators are the products of the blocks that one passes through when going directly from input to output (ignoring the feedback). This type of rule can be used to compute transfer functions by inspection, although for systems with multiple feedback loops it can be tricky to compute them without writing down the algebra explicitly.

We can also use block diagram algebra to obtain insights about state space controllers. Consider a state space controller that uses an observer, such as the one shown in Figure 8.7. The process model is

$$\frac{dx}{dt} = Ax + Bu, \qquad y = Cx,$$

and the controller (8.15) is given by

$$u = -K\hat{x} + k_\mathrm{f} r, \tag{9.32}$$

where $\hat{x}$ is the output of a state observer (8.16) given by

$$\frac{d\hat{x}}{dt} = A\hat{x} + Bu + L(y - C\hat{x}). \tag{9.33}$$

The controller is a system with one output u and two inputs, the reference r and the measured signal y. Using transfer functions and exponential signals it can be represented as

$$u = G_{ur} r + G_{uy} y. \tag{9.34}$$

The transfer function G_{uy} from y to u describes the feedback action and G_{ur} from r to u describes the feedforward action. We call these *open loop* transfer functions because they represent the relationships between the signals without considering the dynamics of the process (e.g., removing P from the system description or cutting the loop at the process input or output).

To derive the controller transfer functions we rewrite equation (9.33) as

$$\frac{d\hat{x}}{dt} = (A - BK - LC)\hat{x} + Bk_\mathrm{f} r + Ly. \tag{9.35}$$

Letting $\hat{x}$, r, and y be exponential signals, the above equations give

$$u = -K\hat{x} + k_\mathrm{f} r, \qquad (sI - (A - BK - LC))\hat{x} = Bk_\mathrm{f} r + Ly,$$

and we find that the controller transfer functions in equation (9.34) are

$$\begin{aligned} G_{ur} &= k_\mathrm{f} - K(sI - A + BK + LC)^{-1} Bk_\mathrm{f}, \\ G_{uy} &= -K(sI - A + BK + LC)^{-1} L. \end{aligned} \tag{9.36}$$

We illustrate with an example.

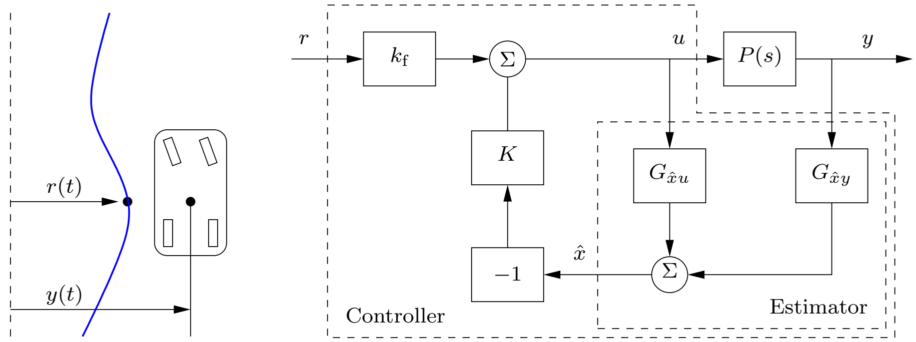


Figure 9.8: Block diagram for a steering control system. The control system is designed to maintain the lateral position of the vehicle along a reference curve (left). The structure of the control system is shown on the right as a block diagram of transfer functions. The estimator consists of two components that compute the estimated state $\hat{x}$ from the combination of the input u and output y of the process. The estimated state is fed through a state feedback controller and combined with a feedforward gain obtain the commanded steering angle u.

Example 9.10 Vehicle steering

Consider the linearized model for vehicle steering introduced in Example 6.13. In Examples 7.4 and 8.3 we designed a state feedback controller and state estimator for the system. A block diagram for the resulting control system is given in Figure 9.8. Note that we have split the estimator into two components, $G_{\hat{x}u}(s)$ and $G_{\hat{x}y}(s)$, corresponding to its inputs u and y. To compute these transfer functions we use equation (9.33) and the expressions for A, B, C, and L from Example 8.3, hence

$$G_{\hat{x}u}(s) = \begin{pmatrix} \dfrac{\gamma s + 1}{s^2 + l_1 s + l_2} \\[2ex] \dfrac{s + l_1 - \gamma l_2}{s^2 + l_1 s + l_2} \end{pmatrix}, \qquad G_{\hat{x}y}(s) = \begin{pmatrix} \dfrac{l_1 s + l_2}{s^2 + l_1 s + l_2} \\[2ex] \dfrac{l_2 s}{s^2 + l_1 s + l_2} \end{pmatrix},$$

where l_1 and l_2 are the observer gains and γ is the scaled position of the center of mass from the rear wheels. Applying block diagram algebra to the controller in Figure 9.8 we obtain

$$G_{ur}(s) = \frac{k_f}{1 + KG_{\hat{x}u}(s)} = \frac{k_f(s^2 + l_1 s + l_2)}{s^2 + s(\gamma k_1 + k_2 + l_1) + k_1 + l_2 + k_2 l_1 - \gamma k_2 l_2},$$

and

$$G_{uy}(s) = \frac{-KG_{\hat{x}y}(s)}{1 + KG_{\hat{x}u}(s)} = \frac{s(k_1 l_1 + k_2 l_2) + k_1 l_2}{s^2 + s(\gamma k_1 + k_2 + l_1) + k_1 + l_2 + k_2 l_1 - \gamma k_2 l_2},$$

where k_1 and k_2 are the state feedback gains and k_f is the feedforward gain. The last equalities are obtained applying block diagram algebra to Figure 9.8, but can also be obtained by applying equation (9.36).

To compute the closed loop transfer function G_{yr} from reference r to output y, we begin by deriving the transfer function for the process $P(s)$. We can compute this directly from the state space description, which was given in Example 6.13. Using that description, we have

$$P(s) = G_{yu}(s) = C(sI - A)^{-1}B + D = \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} s & -1 \\ 0 & s \end{pmatrix}^{-1} \begin{pmatrix} \gamma \\ 1 \end{pmatrix} = \frac{\gamma s + 1}{s^2}.$$

The transfer function for the full closed loop system between the input r and the output y is then given by

$$G_{yr} = \frac{P(s)G_{ur}(s)}{1 - P(s)G_{uy}(s)} = \frac{k_\mathrm{f}(\gamma s + 1)}{s^2 + (k_1\gamma + k_2)s + k_1}.$$

(The unusual sign in the denominator of the middle expression occurs because G_{ur} is in the feedback path and incorporates the -1 gain element.) ∇

Note that in the previous example the observer gains l_1 and l_2 do not appear in the transfer function G_{yr}. This is true in general, as follows from Figure 8.9b in Section 8.3.

We also note that a control system using an observer should be implemented as the multivariable system (9.35), which is of order n. It should not be implemented using two separate transfer functions, as described in equation (9.34), because the controller would then be of order $2n$, and there will be unobservable modes.

Algebraic Loops

When analyzing or simulating a system described by a block diagram, it is necessary to form the differential equations that describe the complete system. In many cases the equations can be obtained by combining the differential equations that describe each subsystem and substituting variables. This simple procedure cannot be used when there are closed loops of subsystems that all have a direct connection between inputs and outputs, known as an *algebraic loop*.

To see what can happen, consider a system with two blocks, a first-order nonlinear system,

$$\frac{dx}{dt} = f(x, u), \qquad y = h(x), \tag{9.37}$$

and a proportional controller described by $u = -ky$. There is no direct term since the function h does not depend on u. In that case we can obtain the equation for the closed loop system simply by replacing u by $-ky$ in equation (9.37) to give

$$\frac{dx}{dt} = f(x, -ky), \qquad y = h(x).$$

Such a procedure can easily be automated using simple formula manipulation.

The situation is more complicated if there is a direct term. If $y = h(x, u)$, then replacing u by $-ky$ gives

$$\frac{dx}{dt} = f(x, -ky), \qquad y = h(x, -ky).$$

To obtain a differential equation for x, the algebraic equation $y = h(x, -ky)$ must be solved to give $y = \alpha(x)$, which in general is a complicated task.

When algebraic loops are present, it is necessary to solve algebraic equations to obtain the differential equations for the complete system. Resolving algebraic loops is a nontrivial problem because it requires the symbolic solution of algebraic equations. Most block diagram-oriented modeling languages cannot handle algebraic loops, and they simply give a diagnosis that such loops are present. In the era of analog computing, algebraic loops were eliminated by introducing fast dynamics between the loops. This created differential equations with fast and slow modes that are difficult to solve numerically. Advanced modeling languages like Modelica use several sophisticated methods to resolve algebraic loops.

9.5 Zero Frequency Gain, Poles, and Zeros

The transfer function has many useful interpretations and the features of a transfer function are often associated with important system properties. Three of the most important features are the gain and the locations of the poles and zeros.

Zero Frequency Gain

The *zero frequency gain* of a system is given by the magnitude of the transfer function at $s = 0$. It represents the ratio of the steady-state value of the output with respect to a step input (which can be represented as $u = e^{st}$ with $s = 0$). For a state space system, we computed the zero frequency gain in equation (6.22):

$$G(0) = D - CA^{-1}B.$$

For a system modeled as the linear differential equation

$$\frac{d^n y}{dt^n} + a_1 \frac{d^{n-1}y}{dt^{n-1}} + \cdots + a_n y = b_0 \frac{d^m u}{dt^m} + b_1 \frac{d^{m-1}u}{dt^{m-1}} + \cdots + b_m u,$$

if we assume that the input u and output y are constants y_0 and u_0, then we find that $a_n y_0 = b_m u_0$, and the zero frequency gain is

$$G(0) = \frac{y_0}{u_0} = \frac{b_m}{a_n}. \tag{9.38}$$

Poles and Zeros

Next consider a linear system with the rational transfer function

$$G(s) = \frac{b(s)}{a(s)}.$$

The roots of the polynomial $a(s)$ are called the *poles* of the system, and the roots of $b(s)$ are called the *zeros* of the system. If p is a pole, it follows that $y(t) = e^{pt}$ is a solution of equation (9.13) with $u = 0$ (the solution to the homogeneous equation). A pole p corresponds to a *mode* of the system with corresponding modal solution e^{pt}. The unforced motion of the system after an arbitrary excitation is a weighted sum of modes.

Zeros have a different interpretation. Since the pure exponential output corresponding to the input $u(t) = e^{st}$ with $a(s) \neq 0$ is $G(s)e^{st}$, it follows that the pure exponential output is zero if $b(s) = 0$. Zeros of the transfer function thus *block transmission* of the corresponding exponential signals.

The difference between the number of poles and zeros $n_{\text{pe}} = n - m$ is called the *pole excess* (also sometimes referred to as the *relative degree*). A rational transfer function is called *proper* if $n_{\text{pe}} \geq 0$ and *strictly proper* if $n_{\text{pe}} > 0$.

Effective use of zeros can be seen in integral control. To obtain a closed loop system where a constant disturbance does not create a steady-state error, the controller is designed so that the transfer function from disturbance to control error has a zero at the origin. Vibration dampers are another example where the system is designed so that the transfer function from disturbance force to motion has a zero at the frequency we want to damp (Example 9.4).

For a state space system with transfer function $G(s) = C(sI - A)^{-1}B + D$, the poles of the transfer function are the eigenvalues of the matrix A in the state space model. One easy way to see this is to notice that the value of $G(s)$ is unbounded when s is an eigenvalue of a system since this is precisely the set of points where the characteristic polynomial $\lambda(s) = \det(sI - A) = 0$ (and hence $sI - A$ is noninvertible). It follows that the poles of a state space system depend only on the matrix A, which represents the intrinsic dynamics of the system. We say that a transfer function is *stable* if all of its poles have negative real part.

To find the zeros of a state space system, we observe that the zeros are complex numbers s such that the input $u(t) = U_0 e^{st}$ gives zero output. Inserting the pure exponential response $x(t) = X_0 e^{st}$ and setting $y(t) = 0$ in equation (9.2) gives

$$se^{st}x_0 = AX_0e^{st} + BU_0e^{st} \qquad 0 = Ce^{st}X_0 + De^{st}U_0,$$

which can be written as

$$\begin{pmatrix} A - sI & B \\ C & D \end{pmatrix} \begin{pmatrix} X_0 \\ U_0 \end{pmatrix} e^{st} = 0.$$

This equation has a solution with nonzero X_0, U_0 only if the matrix on the left does not have full column rank. The zeros are thus the values s such that the matrix

$$\begin{pmatrix} A - sI & B \\ C & D \end{pmatrix} \tag{9.39}$$

loses rank.

Since the zeros depend on A, B, C, and D, they therefore depend on how the inputs and outputs are coupled to the states. Notice in particular that if the matrix B has full row rank, then the matrix in equation (9.39) has n linearly independent rows for all values of s. Similarly there are n linearly independent columns if the matrix C has full column rank. This implies that systems where the matrix B or C is square and full rank do not have zeros. In particular it means that a system has no zeros if it is fully actuated (each state can be controlled independently) or if the full state is measured.

A convenient way to view the poles and zeros of a transfer function is through a *pole zero diagram*, as shown in Figure 9.9. In this diagram, each pole is marked with a cross, and each zero with a circle. If there are multiple poles or zeros at

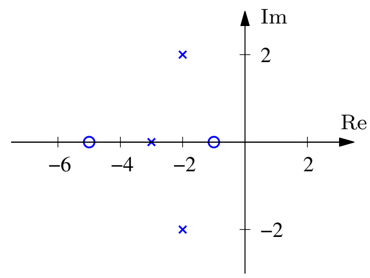


Figure 9.9: A pole zero diagram for a transfer function with zeros at -5 and -1 and poles at -3 and $-2 \pm 2j$. The circles represent the locations of the zeros, and the crosses the locations of the poles.

a fixed location, these are often indicated with overlapping crosses or circles (or other annotations). Poles in the left half-plane correspond to stable modes of the system, and poles in the right half-plane correspond to unstable modes. We thus call a pole in the left half-plane a *stable pole* and a pole in the right half-plane an *unstable pole*. A similar terminology is used for zeros, even though the zeros do not directly relate to stability or instability of the system. Notice that the gain must also be given to have a complete description of the transfer function.

Example 9.11 Balance system
Consider the dynamics for a balance system, shown in Figure 9.10. The transfer function for a balance system can be derived directly from the second-order equations, given in Example 3.2:

$$M_{\mathrm{t}}\frac{d^2q}{dt^2} - ml\frac{d^2\theta}{dt^2}\cos\theta + c\frac{dq}{dt} + ml\sin\theta\big(\frac{d\theta}{dt}\big)^2 = F,$$
$$-ml\cos\theta\frac{d^2q}{dt^2} + J_{\mathrm{t}}\frac{d^2\theta}{dt^2} + \gamma\frac{d\theta}{dt} - mgl\sin\theta = 0.$$

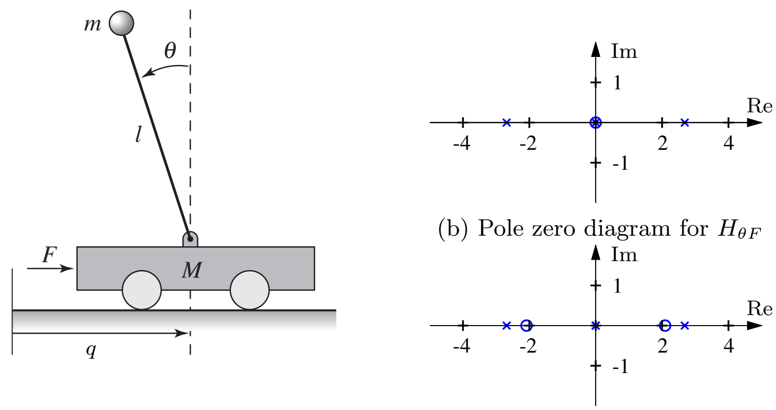
If we assume that θ and $\dot\theta$ are small, we can approximate this nonlinear system by a set of linear second-order differential equations,

$$M_{\mathrm{t}}\frac{d^2q}{dt^2} - ml\frac{d^2\theta}{dt^2} + c\frac{dq}{dt} = F,$$
$$-ml\frac{d^2q}{dt^2} + J_{\mathrm{t}}\frac{d^2\theta}{dt^2} + \gamma\frac{d\theta}{dt} - mgl\theta = 0.$$

If we let F be an exponential signal, the resulting response satisfies

$$M_{\mathrm{t}}s^2\,q - mls^2\,\theta + cs\,q = F,$$
$$J_{\mathrm{t}}s^2\,\theta - mls^2\,q + \gamma s\,\theta - mgl\,\theta = 0,$$

where all signals are exponential signals. The resulting transfer functions for the position of the cart and the orientation of the pendulum are given by solving for q



(a) Cart–pendulum system (b) Pole zero diagram for $H_{\theta F}$ (c) Pole zero diagram for H_{qF}

Figure 9.10: Poles and zeros for a balance system. The balance system (a) can be modeled around its vertical equilibrium point by a fourth order linear system. The poles and zeros for the transfer functions $H_{\theta F}$ and H_{qF} are shown in (b) and (c), respectively.

and θ in terms of F to obtain

$$H_{\theta F}(s) = \frac{mls}{(M_t J_t - m^2l^2)s^3 + (\gamma M_t + cJ_t)s^2 + (c\gamma - M_t mgl)s - mglc},$$
$$H_{qF}(s) = \frac{J_t s^2 + \gamma s - mgl}{(M_t J_t - m^2l^2)s^4 + (\gamma M_t + cJ_t)s^3 + (c\gamma - M_t mgl)s^2 - mglcs},$$

where each of the coefficients is positive. The pole zero diagrams for these two transfer functions are shown in Figure 9.10 using the parameters from Example 7.7.

If we assume the damping is small and set $c = 0$ and $\gamma = 0$, we obtain

$$H_{\theta F}(s) = \frac{ml}{(M_t J_t - m^2l^2)s^2 - M_t mgl},$$
$$H_{qF}(s) = \frac{J_t s^2 - mgl}{s^2\big((M_t J_t - m^2l^2)s^2 - M_t mgl\big)}.$$

This gives nonzero poles and zeros at

$$p = \pm\sqrt{\frac{mglM_t}{M_t J_t - m^2l^2}} \approx \pm 2.68, \qquad z = \pm\sqrt{\frac{mgl}{J_t}} \approx \pm 2.09.$$

We see that these are quite close to the pole and zero locations in Figure 9.10. ∇

Pole/Zero Cancellations

Because transfer functions are often polynomials in s, it can sometimes happen that the numerator and denominator have a common factor, which can be canceled. Sometimes these cancellations are simply algebraic simplifications, but in other situations they can mask potential fragilities in the model. In particular, if a pole/zero cancellation occurs because terms in separate blocks just happen to coincide, the cancellation may not occur if one of the systems is slightly perturbed. In some situations this can result in severe differences between the expected behavior and the actual behavior.

Consider the block diagram in Figure 9.6 with $F = 1$ (no feedforward compensation) and let C and P be given by

$$C(s) = \frac{n_\mathrm{c}(s)}{d_\mathrm{c}(s)}, \qquad P(s) = \frac{n_\mathrm{p}(s)}{d_\mathrm{p}(s)}.$$

The transfer function from r to e is then given by

$$G_{er}(s) = \frac{1}{1+PC} = \frac{d_\mathrm{c}(s)d_\mathrm{p}(s)}{d_\mathrm{c}(s)d_\mathrm{p}(s) + n_\mathrm{c}(s)n_\mathrm{p}(s)}.$$

If there are common factors in the numerator and denominator polynomials, then these terms can be factored out and eliminated from both the numerator and denominator. For example, if the controller has a zero at $s = -a$ and the process has a pole at $s = -a$, then we will have

$$G_{er}(s) = \frac{(s+a)d_\mathrm{c}(s)d'_\mathrm{p}(s)}{(s+a)d_\mathrm{c}(s)d'_\mathrm{p}(s) + (s+a)n'_\mathrm{c}(s)n_\mathrm{p}(s)} = \frac{d_\mathrm{c}(s)d'_\mathrm{p}(s)}{d_\mathrm{c}(s)d'_\mathrm{p}(s) + n'_\mathrm{c}(s)n_\mathrm{p}(s)},$$

where $n'_\mathrm{c}(s)$ and $d'_\mathrm{p}(s)$ represent the relevant polynomials with the term $s + a$ factored out. We see that the $s + a$ term does not appear in the transfer function G_{er}.

Suppose instead that we compute the transfer function from v to e, which represents the effect of a disturbance on the error between the reference and the output. This transfer function is given by

$$G_{ev}(s) = -\frac{d_\mathrm{c}(s)n_\mathrm{p}(s)}{(s+a)d_\mathrm{c}(s)d'_\mathrm{p}(s) + (s+a)n'_\mathrm{c}(s)n_\mathrm{p}(s)}.$$

Notice that if $a < 0$, then the pole is in the right half-plane and the transfer function G_{ev} is *unstable*. Hence, even though the transfer function from r to e appears to be okay (assuming a perfect pole/zero cancellation), the transfer function from v to e can exhibit unbounded behavior. This unwanted behavior is typical of an *unstable pole/zero cancellation*.

As noted at the end of Section 9.2, the cancellation of a pole with a zero can be understood in terms of the state space representation of the systems. Reachability or observability is lost when there are cancellations of poles and zeros (Example 9.7 and Exercise 9.14) and the transfer function depends only on the dynamics in the reachable and observable subsystem Σ_ro.

Example 9.12 Cruise control

A cruise control system can be modeled by the block diagram in Figure 9.6, where y is the vehicle velocity, r the desired velocity, v the slope of the road, and u the throttle. Furthermore $F(s) = 1$, and the input/output response from throttle to

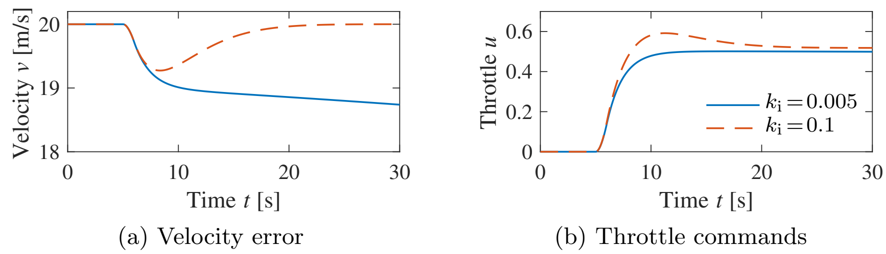


Figure 9.11: Car with PI cruise control encountering a sloping road. The velocity error is shown on the left and the throttle is shown on the right. Results for a PI controller with $k_p = 0.5$ and $k_i = 0.005$ are shown by solid lines, and for a controller with $k_p = 0.5$ and $k_i = 0.1$ are shown by dashed lines. Compare with Figure 4.3b.

velocity for the linearized model for a car has the transfer function $P(s) = b/(s+a)$. A simple (but not necessarily good) way to design a PI controller is to choose the parameters of the PI controller as $k_i = ak_p$. The controller transfer function is then $C(s) = k_p + k_i/s = k_p(s+a)/s$. It has a zero at $s = -k_i/k_p = -a$ that cancels the process pole at $s = -a$. We have $P(s)C(s) = bk_p/s$ giving the transfer function from reference to vehicle velocity as $G_{yr}(s) = bk_p/(s + bk_p)$, and control design is then simply a matter of choosing the gain k_p. The closed loop system dynamics are of first order with the time constant $1/(bk_p)$. Notice that the canceled pole $1/a$ is much slower than the other pole.

Figure 9.11 shows the velocity error when the car encounters an increase in the road slope. A comparison with the controller used in Figure 4.3b (reproduced in dashed curves) shows that the controller based on pole/zero cancellation has very poor performance. The velocity error is larger, and it takes a long time to settle.

Notice that the control signal remains practically constant after $t = 15$ even if the error is large after that time. To understand what happens we will analyze the system. The parameters of the system are $a = 0.01$ and $b = 1.32$, and the controller parameters are $k_p = 0.5$ and $k_i = 0.005$. The closed loop time constant is $1/(bk_p) = 1.5$ s, and we would expect that the error would settle in about 6 s (4 time constants). The transfer functions from road slope to velocity and control signals are

$$G_{yv}(s) = \frac{b_g s}{(s+a)(s+bk_p)}, \qquad G_{uv}(s) = \frac{bk_p}{s+bk_p}.$$

Notice that the slow canceled mode $s = -a = -0.01$ appears in G_{yv} but not in G_{uv}. The reason why the control signal remains constant is that the controller has a zero at $s = -0.01$, which cancels the slowly decaying process mode. Note also that the error would diverge if the canceled pole was unstable. ∇

The lesson we can learn from this example is that it is a bad idea to try to cancel unstable or slow process poles. A more detailed discussion of pole/zero cancellations and their impact on robustness is given in Section 14.5.

9.6 The Bode Plot

The frequency response of a linear system can be computed from its transfer function by setting $s = i\omega$, corresponding to a complex exponential

$$u(t) = e^{i\omega t} = \cos(\omega t) + i\sin(\omega t).$$

The resulting output has the form

$$y(t) = G(i\omega)e^{i\omega t} = Me^{i(\omega t+\theta)} = M\cos(\omega t+\theta) + iM\sin(\omega t+\theta),$$

where M and θ are the gain and phase of G:

$$M = |G(i\omega)|, \qquad \theta = \arctan\frac{\operatorname{Im} G(i\omega)}{\operatorname{Re} G(i\omega)}.$$

The gain and phase of G are also called the *magnitude* and *argument* of G, terms that come from the theory of complex variables.

It follows from linearity that the response to a single sinusoid ($\sin(\omega t)$ or $\cos(\omega t)$) is amplified by M and phase-shifted by θ. It will often be convenient to represent the phase in degrees rather than radians. We will use the notation $\angle G(i\omega)$ for the phase in degrees and $\arg G(i\omega)$ for the phase in radians. In addition, while we always take $\arg G(i\omega)$ to be in the range $(-\pi, \pi]$, we will take $\angle G(i\omega)$ to be continuous, so that it can take on values outside the range of $-180°$ to $180°$.

The frequency response $G(i\omega)$ can thus be represented by two curves: the gain curve and the phase curve. The *gain curve* gives $|G(i\omega)|$ as a function of frequency ω and the *phase curve* gives $\angle G(i\omega)$. One particularly useful way of drawing these curves is to use a log/log scale for the gain curve and a log/linear scale for the phase curve. This type of plot is called a *Bode plot* and is shown in Figure 9.12.

Sketching and Interpreting Bode Plots

Part of the popularity of Bode plots is that they are easy to sketch and interpret. Since the frequency scale is logarithmic, they cover the behavior of a linear system over a wide frequency range.

Consider a transfer function that is a rational function of the form

$$G(s) = \frac{b_1(s)b_2(s)}{a_1(s)a_2(s)}.$$

We have

$$\log|G(s)| = \log|b_1(s)| + \log|b_2(s)| - \log|a_1(s)| - \log|a_2(s)|,$$

and hence we can compute the gain curve by simply adding and subtracting gains corresponding to terms in the numerator and denominator. Similarly,

$$\angle G(s) = \angle b_1(s) + \angle b_2(s) - \angle a_1(s) - \angle a_2(s),$$

and so the phase curve can be determined in an analogous fashion. Since a polynomial can be written as a product of terms of the type

$$k, \quad s, \quad s+a, \quad s^2 + 2\zeta\omega_0 s + \omega_0^2,$$

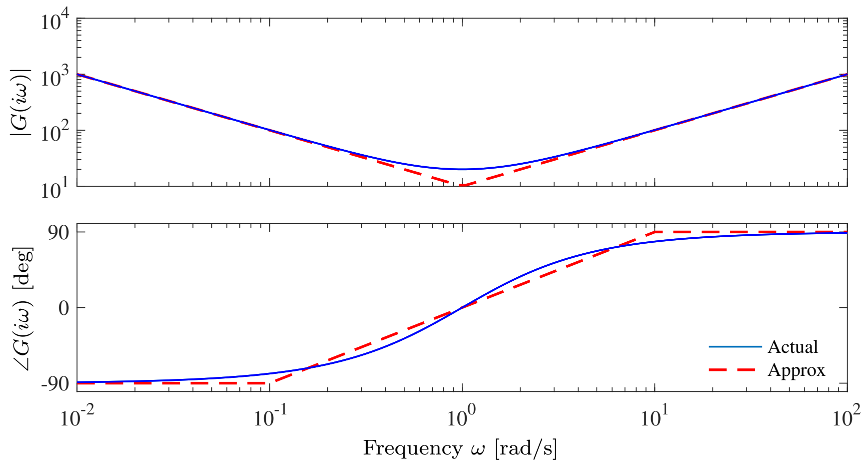


Figure 9.12: Bode plot of the transfer function $C(s) = 20 + \frac{10}{s} + 10s = 10\frac{(s+1)^2}{s}$, corresponding to an ideal PID controller. The upper plot is the gain curve and the lower plot is the phase curve. The dashed lines show straight-line approximations of the gain curve and the corresponding phase curve.

it suffices to be able to sketch Bode diagrams for these terms. The Bode plot of a complex system is then obtained by adding the gains and phases of the terms.

The function $G(s) = s^k$ is a simple transfer function, with the important special cases of $k = 1$ corresponding to a differentiator and $k = -1$ to an integrator. The gain and phase of the term are given by

$$\log|G(i\omega)| = k \times \log\omega, \qquad \angle G(i\omega) = k \times 90°.$$

The gain curve is thus a straight line with slope k, and the phase curve is a constant at $k \times 90°$. The case when $k = 1$ corresponds to a differentiator and has slope 1 with phase 90°. The case when $k = -1$ corresponds to an integrator and has slope -1 with phase $-90°$. Bode plots of the various powers of k are shown in Figure 9.13.

Consider next the transfer function of a first-order system, given by

$$G(s) = \frac{a}{s+a}, \qquad a > 0.$$

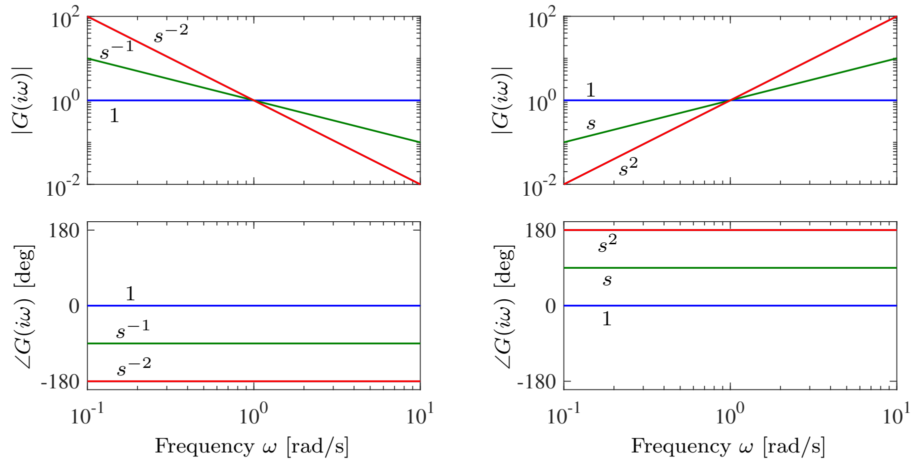
We have

$$|G(s)| = \frac{|a|}{|s+a|}, \qquad \angle G(s) = \angle(a) - \angle(s+a),$$

and hence

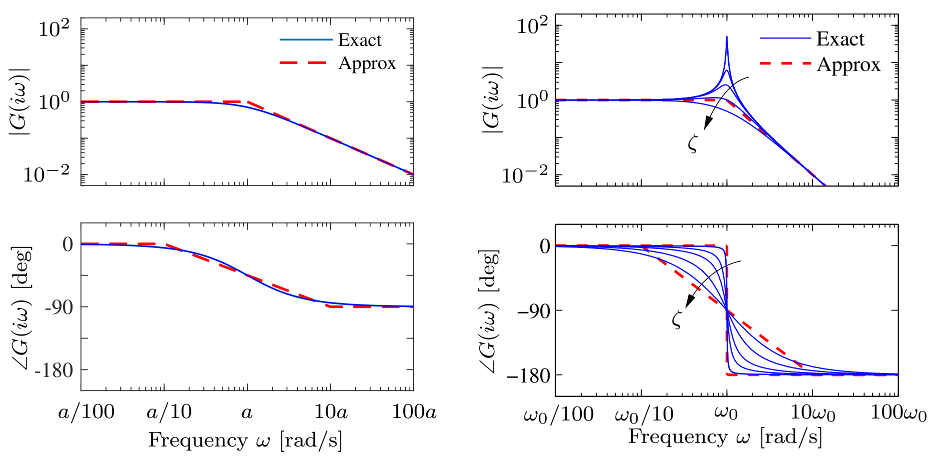
$$\log|G(i\omega)| = \log a - \frac{1}{2}\log(\omega^2 + a^2), \qquad \angle G(i\omega) = -\frac{180}{\pi}\arctan\frac{\omega}{a}.$$

The Bode plot is shown in Figure 9.14a, with the magnitude normalized by the zero frequency gain. Both the gain curve and the phase curve can be approximated by



(a) s^k, $k < 0$ (b) s^k, $k > 0$

Figure 9.13: Bode plots of the transfer functions $G(s) = s^k$ for $k = -2, -1, 0, 1, 2$. On a log-log scale, the gain curve is a straight line with slope k. The phase curves for the transfer functions are constants, with phase equal to $k \times 90°$.



(a) First-order system (b) Second-order system

Figure 9.14: Bode plots for first- and second-order systems. (a) The first-order system $G(s) = a/(s + a)$ can be approximated by asymptotic curves (dashed) in both the gain and the frequency, with the breakpoint in the gain curve at $\omega = a$ and the phase decreasing by 90° over a factor of 100 in frequency. (b) The second-order system $G(s) = \omega_0^2/(s^2 + 2\zeta\omega_0 s + \omega_0^2)$ has a peak at frequency ω_0 and then a slope of -2 beyond the peak; the phase decreases from 0° to $-180°$. The height of the peak and the rate of change of phase depending on the damping ratio ζ ($\zeta = 0.02$, 0.1, 0.2, 0.5, and 1.0 shown).

the following straight lines

$$\log|G(i\omega)| \approx \begin{cases} 0 & \text{if } \omega < a, \\ \log a - \log\omega & \text{if } \omega > a, \end{cases}$$

$$\angle G(i\omega) \approx \begin{cases} 0 & \text{if } \omega < a/10, \\ -45 - 45(\log\omega - \log a,) & \text{if } a/10 < \omega < 10a, \\ -90 & \text{if } \omega > 10a. \end{cases}$$

The approximate gain curve consists of a horizontal line up to frequency $\omega = a$, called the *breakpoint* or *corner frequency*, after which the curve is a line of slope -1 (on a log-log scale). The phase curve is zero up to frequency $a/10$ and then decreases linearly by 45°/decade up to frequency $10a$, at which point it remains constant at $-90°$. Notice that a first-order system behaves like a constant for low frequencies and like an integrator for high frequencies; compare with the Bode plot in Figure 9.13.

Finally, consider the transfer function for a second-order system,

$$G(s) = \frac{\omega_0^2}{s^2 + 2\omega_0\zeta s + \omega_0^2},$$

with $0 < \zeta < 1$, for which we have

$$\log|G(i\omega)| = 2\log\omega_0 - \frac{1}{2}\log\left(\omega^4 + 2\omega_0^2\omega^2(2\zeta^2 - 1) + \omega_0^4\right),$$

$$\angle G(i\omega) = -\frac{180}{\pi}\arctan\frac{2\zeta\omega_0\omega}{\omega_0^2 - \omega^2}.$$

The gain curve has an asymptote with zero slope for $\omega \ll \omega_0$. For large values of ω the gain curve has an asymptote with slope -2. The largest gain $Q = \max_\omega |G(i\omega)| \approx 1/(2\zeta)$, called the *Q-value*, is obtained for $\omega \approx \omega_0$. The phase is zero for low frequencies and approaches $-180°$ for large frequencies. The curves can be approximated with the following piecewise linear expressions

$$\log|G(i\omega)| \approx \begin{cases} 0 & \text{if } \omega \ll \omega_0, \\ 2\log\omega_0 - 2\log\omega & \text{if } \omega \gg \omega_0, \end{cases}$$

$$\angle G(i\omega) \approx \begin{cases} 0 & \text{if } \omega \ll \omega_0, \\ -180 & \text{if } \omega \gg \omega_0. \end{cases}$$

The Bode plot is shown in Figure 9.14b. Note that the asymptotic approximation is poor near $\omega = \omega_0$ and that the Bode plot depends strongly on ζ near this frequency.

Given the Bode plots of the basic functions, we can now sketch the frequency response for a more general system. The following example illustrates the basic idea.

Example 9.13 Asymptotic approximation for a transfer function
Consider the transfer function given by

$$G(s) = \frac{k(s+b)}{(s+a)(s^2 + 2\zeta\omega_0 s + \omega_0^2)}, \qquad a \ll b \ll \omega_0.$$

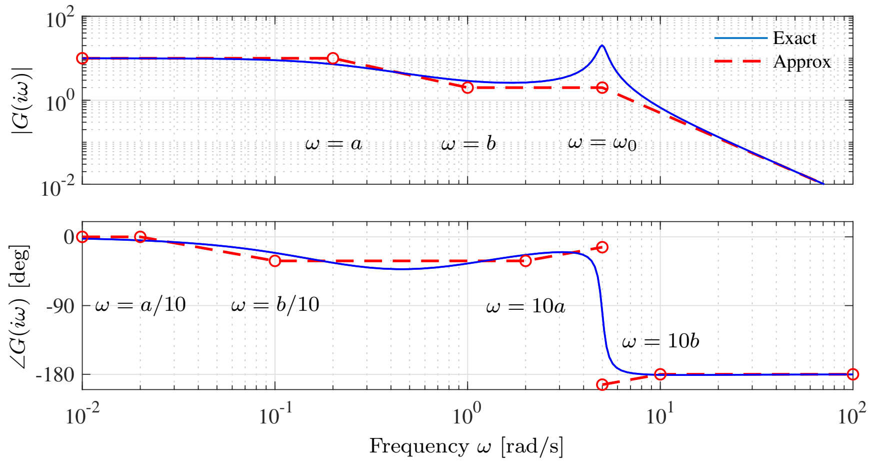


Figure 9.15: Asymptotic approximation to a Bode plot. The solid curve is the Bode plot for the transfer function $G(s) = k(s+b)/(s+a)(s^2+2\zeta\omega_0 s+\omega_0^2)$, where $a \ll b \ll \omega_0$. Each segment in the gain and phase curves represents a separate portion of the approximation, where either a pole or a zero begins to have effect. Each segment of the approximation is a straight line between these points at a slope given by the rules for computing the effects of poles and zeros.

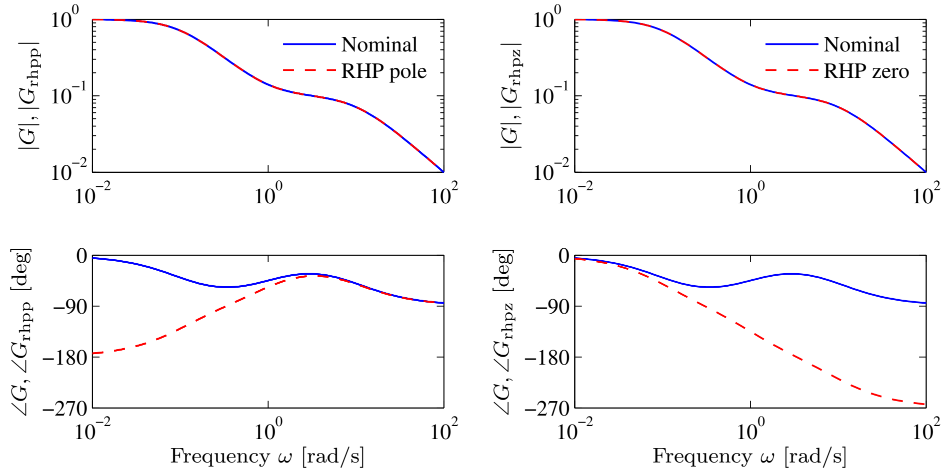
The Bode plot for this transfer function appears in Figure 9.15, with the complete transfer function shown as a solid curve and the asymptotic approximation shown as a dashed curve.

We begin with the gain curve. At low frequency, the magnitude is given by

$$G(0) = \frac{kb}{a\omega_0^2}.$$

When we reach $\omega = a$, the effect of the pole begins and the gain decreases with slope -1. At $\omega = b$, the zero comes into play and we increase the slope by 1, leaving the asymptote with net slope 0. This slope is used until the effect of the second-order pole is seen at $\omega = \omega_0$, at which point the asymptote changes to slope -2. We see that the gain curve is fairly accurate except in the region of the peak due to the second-order pole (indicating that for this case ζ is reasonably small).

The phase curve is more complicated since the effect of the phase stretches out much further. The effect of the pole begins at $\omega = a/10$, at which point we change from phase 0 to a slope of $-45°$/decade. The zero begins to affect the phase at $\omega = b/10$, producing a flat section in the phase. At $\omega = 10a$ the phase contribution from the pole ends, and we are left with a slope of $+45°$/decade (from the zero). At the location of the second-order pole, $s \approx i\omega_0$, we get a jump in phase of $-180°$. Finally, at $\omega = 10b$ the phase contribution of the zero ends, and we are left with a phase of -180 degrees. We see that the straight-line approximation for the phase is not quite as accurate as it was for the gain curve, but it does capture the basic features of the phase changes as a function of frequency. ∇



(a) Right half-plane pole (b) Right half-plane zero

Figure 9.16: Effect of a right half-plane pole and a right half-plane zero on the Bode plot. The curves for G, which has all poles and zeros in the right half-plane, are shown in solid lines and the curves for G_{rhpp} and G_{rhpz} are shown as dashed curves. (a) Bode plots for the transfer functions G and G_{rhpp}, which have a pole at $s = -10$ and a zero at $s = -1$, but G has a pole at $s = -0.1$ while G_{rhpp} has a corresponding pole at $s = 0.1$. (b) Bode plots for the transfer functions G and G_{rhpz}, which have the same poles at $s- = 0.1$ and $s = -10$, while G has a zero at $s = -1$ and G_{rhpz} has a zero $s = 1$.

Poles and Zeros in the Right Half-Plane

The gain curve of a transfer function remains the same if a pole or a zero of a transfer function is shifted from the left half-plane to the right half-plane by mirror imaging in the imaginary axis. The phase will, however, change significantly as is illustrated by the following example.

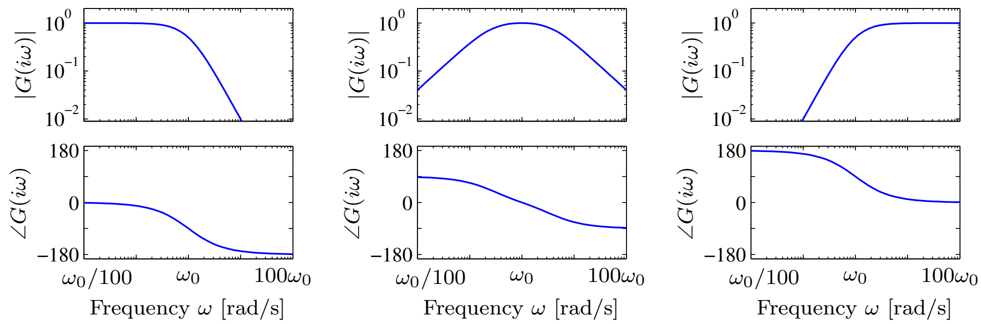
Example 9.14 Transfer function with a zero in the right half-plane
Consider the transfer functions

$$G(s) = \frac{s+1}{(s+0.1)(s+10)}, \qquad G_{\mathrm{rhpp}}(s) = \frac{s+1}{(s-0.1)(s+10)},$$

and

$$G_{\mathrm{rhpz}}(s) = \frac{-s+1}{(s+0.1)(s+10)}.$$

The transfer functions G and G_{rhpp} have the zero at $s = -1$ and the pole at $s = -10$ in common, while G has the pole at $s = -0.1$ but G_{rhpp} has the pole at $s = 0.1$. Similarly, the transfer functions G and G_{rhpz} have the same poles, but G has the zero at $s = -1$ while G_{rhpz} has the zero at $s = 1$. Notice that all transfer functions have the same gain curves but that the phase curves differ significantly, as shown in Figure 9.16. Notice in particular that the transfer functions G_{rhpp} and G_{rhpz} have much larger phase lags than G. ∇



$G(s) = \dfrac{\omega_0^2}{s^2 + 2\zeta\omega_0 s + \omega_0^2}$

(a) Low-pass filter

$G(s) = \dfrac{2\zeta\omega_0 s}{s^2 + 2\zeta\omega_0 s + \omega_0^2}$

(b) Band-pass filter

$G(s) = \dfrac{s^2}{s^2 + 2\zeta\omega_0 s + \omega_0^2}$

(c) High-pass filter

Figure 9.17: Bode plots for low-pass, band-pass, and high-pass filters. The upper plots are the gain curves and the lower plots are the phase curves. Each system passes frequencies in a different range and attenuates frequencies outside of that range.

A time delay, which has the transfer function $G(s) = e^{-s\tau}$, is an even more striking example of a change in phase than a right half-plane zero. Since $|G(i\omega\tau)| = |e^{-i\omega\tau}| = 1$ the gain curve is constant but the phase is $\angle G(i\omega\tau) = -180\,\omega\tau/\pi$, which has a large negative value for large ω. Time delays are in this respect similar to right half-plane zeros. Intuitively it seems reasonable that extra phase will cause difficulties for control since there is a delay between applying an input and seeing its effect. Poles and zeros in the right half-plane and time delay will indeed limit the achievable control performance, as will be discussed in detail in Section 10.4 and Chapter 14.

System Insights from the Bode Plot

The Bode plot gives a quick overview of a system. The plot covers wide ranges in amplitude and frequency because of the logarithmic scales. Since many useful signals can be decomposed into a sum of sinusoids, it is possible to visualize the behavior of a system for different frequency ranges. The system can be viewed as a filter that can change the amplitude (and phase) of the input signals according to the frequency response. For example, if there are frequency ranges where the gain curve has constant slope and the phase is close to zero, the action of the system for signals with these frequencies can be interpreted as a pure gain. Similarly, for frequencies where the slope is +1 and the phase close to 90°, the action of the system can be interpreted as a differentiator.

Three common types of frequency responses are shown in Figure 9.17. The system in Figure 9.17a is called a *low-pass filter* because the gain is constant for low frequencies and drops for high frequencies. Notice that the phase is zero for low frequencies and −180° for high frequencies. The systems in Figures 9.17b and 9.17c are called a *band-pass filter* and a *high-pass filter* for similar reasons.

To illustrate how different system behaviors can be read from the Bode plots

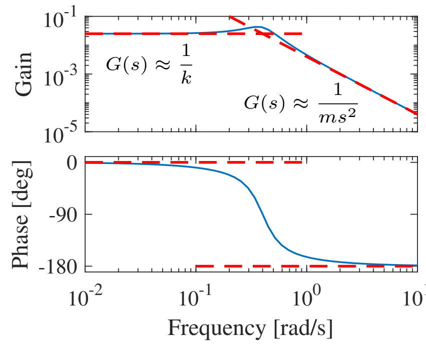


Figure 9.18: Bode plot for a spring–mass system. At low frequency the system behaves like a spring with $G(s) \approx 1/k$ and at high frequency the system behaves like a pure mass with $G(s) \approx 1/(ms^2)$.

we consider the band-pass filter in Figure 9.17b. For frequencies around $\omega = \omega_0$, the signal is passed through with no change in gain. However, for frequencies well below or well above ω_0, the signal is attenuated. The phase of the signal is also affected by the filter, as shown in the phase curve. For frequencies below $\omega_0/100$ there is a phase lead of 90°, and for frequencies above $100\omega_0$ there is a phase lag of 90°. These actions correspond to differentiation and integration of the signal in these frequency ranges.

The intuition captured in the Bode plot can also be related to the transfer function: the approximations of $G(s)$ for small and large s capture the propagation of slow and fast signals, respectively, as illustrated in the following examples.

Example 9.15 Spring–mass system

Consider a spring–mass system with input u (force) and output q (position), whose dynamics satisfy the second-order differential equation

$$m\ddot{q} + c\dot{q} + kq = u.$$

The system has the transfer function

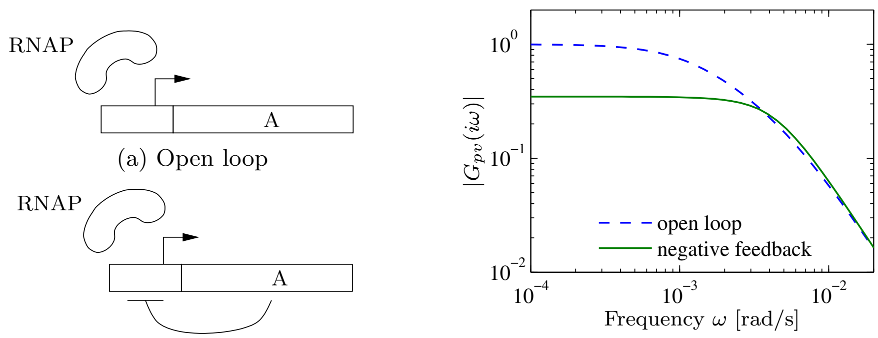
$$G(s) = \frac{1}{ms^2 + cs + k},$$

and the Bode plot is shown in Figure 9.18. For small s we have $G(s) \approx 1/k$. The corresponding input/output relation is $q = (1/k)u$, which implies that for low-frequency inputs, the system behaves like a spring driven by a force. For large s we have $G(s) \approx 1/(ms^2)$. The corresponding differential equation is $m\ddot{q} = u$ and the system thus behaves like mass driven by a force (a double integrator). ∇

Example 9.16 Transcriptional regulation

Consider a genetic circuit consisting of a single gene. We wish to study the response of the protein concentration to fluctuations in the mRNA dynamics. We consider two cases: a “constitutive” promoter (no regulation) and self-repression (negative feedback), illustrated in Figure 9.19. The dynamics of the system are given by

$$\frac{dm}{dt} = \alpha(p) - \delta m + v, \qquad \frac{dp}{dt} = \kappa m - \gamma p,$$



(a) Open loop

(b) Negative feedback

(c) Frequency response

Figure 9.19: Noise attenuation in a genetic circuit. The open loop system (a) consists of a constitutive promoter, while the closed loop circuit (b) is self-regulated with negative feedback (repressor). The frequency response for each circuit is shown in (c).

where v is a disturbance term that affects mRNA transcription.

For the case of no feedback we have $\alpha(p) = \alpha_0$, and when $v = 0$ the system has an equilibrium point at $m_e = \alpha_0/\delta$, $p_e = \kappa\alpha_0/(\gamma\delta)$. The open loop transfer function from v to p is given by

$$G_{pv}^{\mathrm{ol}}(s) = \frac{\kappa}{(s+\delta)(s+\gamma)}.$$

For the case of negative regulation, we have

$$\alpha(p) = \frac{\alpha_1}{1+kp^n} + \alpha_0,$$

and the equilibrium points satisfy

$$m_e = \frac{\gamma}{\kappa} p_e, \qquad \frac{\alpha_1}{1+kp_e^n} + \alpha_0 = \delta m_e = \frac{\delta\gamma}{\kappa} p_e.$$

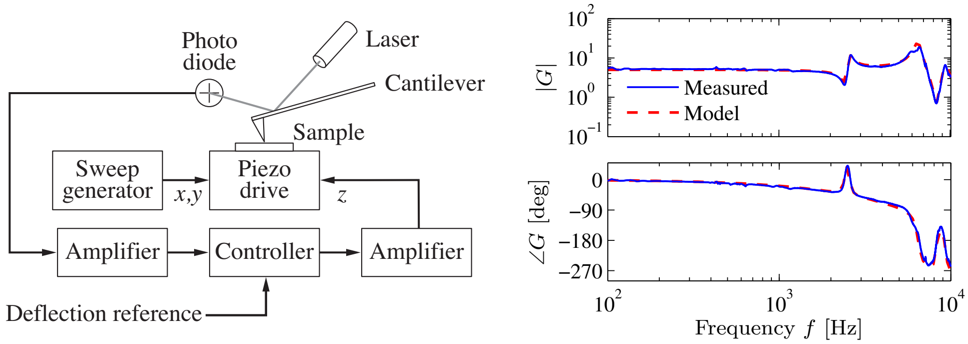
The transfer function can be obtained by linearization around the equilibrium point and can be shown to be

$$G_{pv}^{\mathrm{cl}}(s) = \frac{\kappa}{(s+\delta)(s+\gamma)+\kappa\sigma}, \qquad \sigma = \frac{n\alpha_1 k p_e^{n-1}}{(1+kp_e^n)^2}.$$

Figure 9.19c shows the frequency response for the two circuits. We see that the feedback circuit attenuates the response of the system to disturbances with low-frequency content but slightly amplifies disturbances at high frequency (compared to the open loop system). ∇

Determining Transfer Functions Experimentally

The transfer function of a system provides a summary of the input/output response and is very useful for analysis and design. We can often build an input/output



(a) AFM block diagram (b) Frequency response

Figure 9.20: Frequency response of a preloaded piezoelectric drive for an atomic force microscope. The Bode plot shows the response of the measured transfer function (solid) and the fitted transfer function (dashed).

model for a given application by directly measuring the frequency response and fitting a transfer function to it. To do so, we perturb the input to the system using a sinusoidal signal at a fixed frequency. When steady state is reached, the amplitude ratio and the phase lag give the frequency response for the excitation frequency. The complete frequency response is obtained by sweeping over a range of frequencies.

By using correlation techniques it is possible to determine the frequency response very accurately, and an analytic transfer function can be obtained from the frequency response by curve fitting. The success of this approach has led to instruments and software that automate this process, called *spectrum analyzers*. We illustrate the basic concept through two examples.

Example 9.17 Atomic force microscope

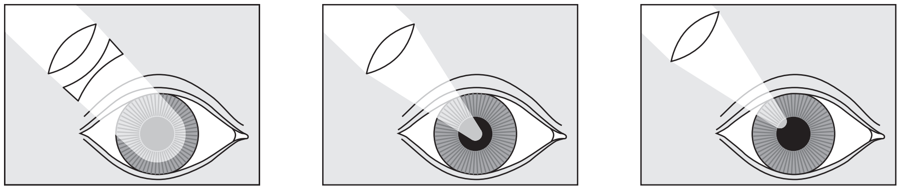
To illustrate the utility of spectrum analysis, we consider the dynamics of the atomic force microscope, described in Section 4.5. Experimental determination of the frequency response is particularly attractive for this system because its dynamics are very fast and hence experiments can be done quickly. A typical example is given in Figure 9.20, which shows an experimentally determined frequency response (solid line). In this case the frequency response was obtained in less than a second. The transfer function

$$G(s) = \frac{k\omega_2^2\omega_3^2\omega_5^2(s^2+2\zeta_1\omega_1 s+\omega_1^2)(s^2+2\zeta_4\omega_4 s+\omega_4^2)e^{-s\tau}}{\omega_1^2\omega_4^2(s^2+2\zeta_2\omega_2 s+\omega_2^2)(s^2+2\zeta_3\omega_3 s+\omega_3^2)(s^2+2\zeta_5\omega_5 s+\omega_5^2)},$$

with $\omega_i = 2\pi f_i$, $k = 5$,

$$\begin{aligned} &f_1 = 2.4\text{ kHz}, && f_2 = 2.6\text{ kHz}, && f_3 = 6.5\text{ kHz}, && f_4 = 8.3\text{ kHz}, && f_5 = 9.3\text{ kHz}, \\ &\zeta_1 = 0.03, && \zeta_2 = 0.03, && \zeta_3 = 0.042, && \zeta_4 = 0.025, && \zeta_5 = 0.032, \end{aligned}$$

and $\tau = 10^{-4}$ s, was fitted to the data (dashed line). The frequencies ω_1 and ω_4 associated with the zeros are located where the gain curve has minima, and the



(a) Closed loop (b) Open loop (c) High gain

Figure 9.21: Light stimulation of the eye. In (a) the light beam is so large that it always covers the whole pupil, giving closed loop dynamics. In (b) the light is focused into a beam which is so narrow that it is not influenced by the pupil opening, giving open loop dynamics. In (c) the light beam is focused on the edge of the pupil opening, which has the effect of increasing the gain of the system since small changes in the pupil opening have a large effect on the amount of light entering the eye. From Stark [Sta68].

frequencies ω_2, ω_3, and ω_5 associated with the poles are located where the gain curve has local maxima. The relative damping ratios are adjusted to give a good fit to maxima and minima. When a good fit to the gain curve is obtained, the time delay is adjusted to give a good fit to the phase curve. The piezo drive is preloaded, and a simple model of its dynamics is derived in Exercise 4.6. The pole at 2.55 kHz corresponds to a "trampoline" mode; the other resonances are higher modes. ∇

Example 9.18 Pupillary light reflex dynamics

The human eye is an organ that is easily accessible for experiments. It has a control system that adjusts the pupil opening to regulate the light intensity at the retina.

This control system was explored extensively by Stark in the 1960s [Sta68]. To determine the dynamics, light intensity on the eye was varied sinusoidally and the pupil opening was measured. A fundamental difficulty is that the closed loop system is insensitive to internal system parameters, so analysis of a closed loop system thus gives little information about the internal properties of the system. Stark used a clever experimental technique that allowed him to investigate both open and closed loop dynamics. He excited the system by varying the intensity of a light beam focused on the eye and measured pupil area, as illustrated in Figure 9.21. By using a wide light beam that covers the whole pupil, the measurement gives the closed loop dynamics. The open loop dynamics were obtained by using a narrow beam, which is small enough that it is not influenced by the pupil opening. The result of one experiment for determining open loop dynamics is given in Figure 9.22. Fitting a transfer function to the gain curve gives a good fit for $G(s) = 0.17/(1 + 0.08s)^3$. This curve gives a poor fit to the phase curve as shown by the dashed curve in Figure 9.22. The fit to the phase curve is improved by adding a 0.2 s time delay, which leaves the gain curve unchanged while substantially modifying the phase curve. The final fit gives the model

$$G(s) = \frac{0.17}{(1 + 0.08s)^3} e^{-0.2s}.$$

The Bode plot of this is shown with solid curves in Figure 9.22. Modeling of the pupillary reflex from first principles is discussed in detail in [KS09]. ∇

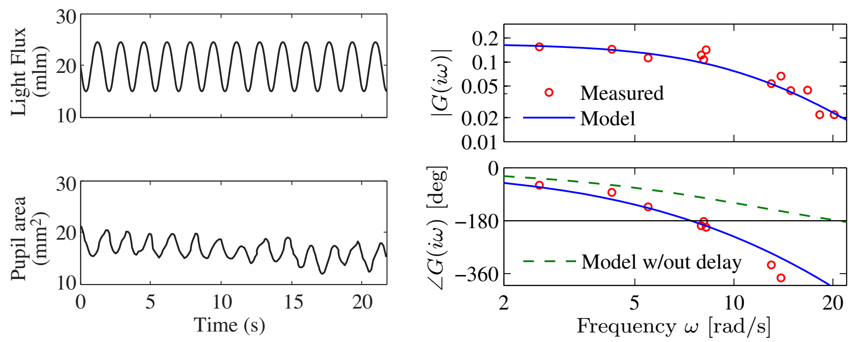


Figure 9.22: Sample curves from an open loop frequency response of the eye (left) and a Bode plot for the open loop dynamics (right). The solid curve shows a fit of the data using a third-order transfer function with 0.2 s time delay. The dashed curve in the Bode plot is the phase of the system without time delay, showing that the delay is needed to properly capture the phase. (Figure redrawn from the data of Stark [Sta68].)

Notice that for both the AFM drive and pupillary dynamics it is not easy to derive appropriate models from first principles. In practice, it is often fruitful to use a combination of analytical modeling and experimental identification of parameters. Experimental determination of frequency response is less attractive for systems with slow dynamics because the experiment takes a long time.

9.7 Further Reading

The idea of characterizing a linear system by its steady-state response to sinusoids was introduced by Fourier in his investigation of heat conduction in solids [Fou07]. Much later, it was used by the electrical engineer Steinmetz who introduced the $i\omega$ method for analyzing electrical circuits. Transfer functions were introduced via the Laplace transform by Gardner and Barnes [GB42], who also used them to calculate the response of linear systems. The Laplace transform was very important in the early phase of control because it made it possible to find transients via tables (see, e.g., [JNP47]). Combined with block diagrams and transfer functions, Laplace transforms provided powerful techniques for dealing with complex systems. Calculation of responses based on Laplace transforms is less important today, when responses of linear systems can easily be generated using computers. The frequency response of a system can also be measured directly using a frequency response analyzer. There are many excellent books on the use of Laplace transforms and transfer functions for modeling and analysis of linear input/output systems. Traditional texts on control such as [DB04], [FPEN05] and [Oga01] are representative examples. Pole/zero cancellation was one of the mysteries of early control theory. It is clear that common factors can be canceled in a rational function, but cancellations have system theoretical consequences that were not clearly understood until Kalman's decomposition of a linear system was introduced [KHN63]. In the following chapters, we will use transfer functions extensively to analyze stability and to describe model uncertainty.

Exercises

9.1 Consider the system

$$\frac{dx}{dt} = ax + u.$$

Compute the exponential response of the system and use this to derive the transfer function from u to x. Show that when $s = a$, a pole of the transfer function, the response to the exponential input $u(t) = e^{st}$ is $x(t) = e^{at}x(0) + te^{at}$.

9.2 Let $G(s)$ be the transfer function for a linear system. Show that if we apply an input $u(t) = A\sin(\omega t)$, then the steady-state output is given by $y(t) = |G(i\omega)|A\sin(\omega t + \arg G(i\omega))$. (Hint: Start by showing that the real part of a complex number is a linear operation and then use this fact.)

9.3 (Inverted pendulum) A model for an inverted pendulum was introduced in Example 3.3. Neglecting damping and linearizing the pendulum around the upright position gives a linear system characterized by the matrices

$$A = \begin{pmatrix} 0 & 1 \\ mgl/J_\mathrm{t} & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ 1/J_\mathrm{t} \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 0 \end{pmatrix}, \quad D = 0.$$

Determine the transfer function of the system.

9.4 (Operational amplifier) Consider the operational amplifier described in Section 4.3 and analyzed in Example 9.2. An analog implementation of a PI controller can be constructed using an op amp by replacing the resistor R_2 with a resistor and capacitor in series, as shown in Figure 4.10. The resulting transfer function of the circuit is given by

$$H(s) = -\left(R_2 + \frac{1}{Cs}\right) \cdot \left(\frac{kCs}{\big((k+1)R_1C + R_2C\big)s + 1}\right),$$

where k is the gain of the op amp, R_1 and R_2 are the resistances in the compensation network and C is the capacitance.

(a) Sketch the Bode plot for the system under the assumption that $k \gg R_2 > R_1$. You should label the key features in your plot, including the gain and phase at low frequency, the slopes of the gain curve, the frequencies at which the gain changes slope, etc.

(b) Suppose now that we include some dynamics in the amplifier, as outlined in Example 9.2. This would involve replacing the gain k with the transfer function

$$G(s) = \frac{ak}{s+a}.$$

Compute the resulting transfer function for the system (i.e., replace k with $G(s)$) and find the poles and zeros assuming the following parameter values

$$\frac{R_2}{R_1} = 100, \qquad k = 10^6, \qquad R_2C = 1, \qquad a = 100.$$

(c) Sketch the Bode plot for the transfer function in part (b) using straight line approximations and compare this to the exact plot of the transfer function (using MATLAB). Make sure to label the important features in your plot.

9.5 (Delay differential equation) Consider a system described by

$$\frac{dx}{dt} = -x(t) + u(t-\tau).$$

Derive the transfer function for the system.

9.6 (Congestion control) Consider the congestion control model described in Section 4.4. Let w represent the individual window size for a set of N identical sources, q represent the end-to-end probability of a dropped packet, b represent the number of packets in the router's buffer, and p represent the probability that a packet is dropped by the router. We write $\bar{w} = Nw$ to represent the total number of packets being received from all N sources. Show that the linearized model can be described by the transfer functions

$$G_{b\bar{w}}(s) = \frac{e^{-\tau^{\mathrm{f}} s}}{\tau_{\mathrm{e}}^{\mathrm{p}} s + e^{-\tau^{\mathrm{f}} s}}, \qquad G_{\bar{w}q}(s) = \frac{N}{q_{\mathrm{e}}(\tau_{\mathrm{e}}^{\mathrm{p}} s + q_{\mathrm{e}} w_{\mathrm{e}})},$$
$$G_{qp}(s) = e^{-\tau^{\mathrm{b}} s}, \qquad G_{pb}(s) = \rho e^{-\tau_{\mathrm{e}}^{\mathrm{p}} s},$$

where $(w_{\mathrm{e}}, b_{\mathrm{e}})$ is the equilibrium point for the system, $\tau_{\mathrm{e}}^{\mathrm{p}}$ is the router processing time, and τ^{f} and τ^{b} are the forward and backward propagation times.

9.7 (Transfer function for state space system) Consider the linear state space system

$$\frac{dx}{dt} = Ax + Bu, \qquad y = Cx.$$

(a) Show that the transfer function is

$$G(s) = \frac{b_1 s^{n-1} + b_2 s^{n-2} + \cdots + b_n}{s^n + a_1 s^{n-1} + \cdots + a_n},$$

where the coefficients for the numerator polynomial are linear combinations of the *Markov parameters* CA^iB, $i = 0, \ldots, n-1$:

$$b_1 = CB, \quad b_2 = CAB + a_1 CB, \quad \ldots, \quad b_n = CA^{n-1}B + a_1 CA^{n-2}B + \cdots + a_{n-1}CB$$

and $\lambda(s) = s^n + a_1 s^{n-1} + \cdots + a_n$ is the characteristic polynomial for A.

(b) Compute the transfer function for a linear system in reachable canonical form and show that it matches the transfer function given above.

9.8 Consider linear time-invariant systems with the control matrices

$$(a) \quad A = \begin{pmatrix} -1 & 0 \\ 0 & -2 \end{pmatrix}, \quad B = \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & -1 \end{pmatrix}, \quad D = 0,$$
$$(b) \quad A = \begin{pmatrix} -3 & 1 \\ -2 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 1 \\ 3 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 0 \end{pmatrix}, \quad D = 0,$$
$$(c) \quad A = \begin{pmatrix} -3 & -2 \\ 1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 3 \end{pmatrix}, \quad D = 0.$$

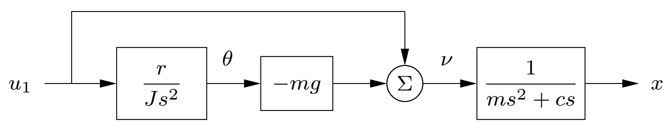
Show that all systems have the transfer function $G(s) = \dfrac{s+3}{(s+1)(s+2)}$.

9.9 (Kalman decomposition) Show that the transfer function of a system depends only on the dynamics in the reachable and observable subspace of the Kalman decomposition. (Hint: Consider the representation given by equation (8.20).)

9.10 Using block diagram algebra, show that the transfer functions from v to y and w to y in Figure 9.6 are given by

$$G_{yv} = \frac{P}{1+PC}, \qquad G_{yw} = \frac{1}{1+PC}.$$

9.11 (Vectored thrust aircraft) Consider the lateral dynamics of a vectored thrust aircraft as described in Example 3.12. Show that the dynamics can be described using the following block diagram:

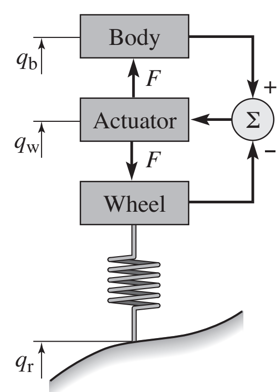


Use this block diagram to compute the transfer functions from u_1 to θ and x and show that they satisfy

$$H_{\theta u_1} = \frac{r}{Js^2}, \qquad H_{xu_1} = \frac{Js^2 - mgr}{Js^2(ms^2+cs)}.$$

9.12 (Vehicle suspension [HB90]) Active and passive damping are used in cars to give a smooth ride on a bumpy road. A schematic diagram of a car with a damping system in shown in the following figure.

(Porter Class I race car driven by Todd Cuffaro)



This model is called a *quarter car model*, and the car is approximated with two masses, one representing one fourth of the car body and the other a wheel. The actuator exerts a force F between the wheel and the body based on feedback from the distance between the body and the center of the wheel (the *rattle space*).

Let q_b, q_w, and q_r represent the heights of body, wheel, and road measured from their equilibrium points. A simple model of the system is given by Newton's equations for the body and the wheel,

$$m_\mathrm{b}\ddot{q}_\mathrm{b} = F, \qquad m_\mathrm{w}\ddot{q}_\mathrm{w} = -F + k_t(q_\mathrm{r} - q_\mathrm{w}),$$

where m_b is a quarter of the body mass, m_w is the effective mass of the wheel including brakes and part of the suspension system (the *unsprung mass*) and k_t is the tire stiffness. For a conventional damper consisting of a spring and a damper, we have $F = k(q_\mathrm{w} - q_\mathrm{b}) + c(\dot{q}_\mathrm{w} - \dot{q}_\mathrm{b})$. For an active damper the force F can be more general and can also depend on riding conditions. Rider comfort can be characterized by the transfer function G_{aq_r} from road height q_r to body acceleration $a = \ddot{q}_\mathrm{b}$. Show that this transfer function has the property $G_{aq_\mathrm{r}}(i\omega_t) = k_t/m_\mathrm{b}$, where $\omega_t = \sqrt{k_t/m_\mathrm{w}}$ (the *tire hop frequency*). The equation implies that there are fundamental limits to the comfort that can be achieved with any damper.

9.13 (Solutions corresponding to poles and zeros) Consider the differential equation

$$\frac{d^n y}{dt^n} + a_1\frac{d^{n-1}y}{dt^{n-1}} + \cdots + a_n y = b_1\frac{d^{n-1}u}{dt^{n-1}} + b_2\frac{d^{n-2}u}{dt^{n-2}} + \cdots + b_n u.$$

(a) Let λ be a root of the characteristic equation

$$s^n + a_1 s^{n-1} + \cdots + a_n = 0.$$

Show that if $u(t) = 0$, the differential equation has the solution $y(t) = e^{\lambda t}$.

(b) Let κ be a zero of the polynomial

$$b(s) = b_1 s^{n-1} + b_2 s^{n-2} + \cdots + b_n.$$

Show that if the input is $u(t) = e^{\kappa t}$, then there is a solution to the differential equation that is identically zero.

9.14 (Pole/zero cancellation) Consider a closed loop system of the form of Figure 9.6, with $F = 1$ and P and C having a pole/zero cancellation. Show that if each system is written in state space form, the resulting closed loop system is not reachable and not observable.

9.15 (Inverted pendulum with PD control) Consider the normalized inverted pendulum system, whose transfer function is given by $P(s) = 1/(s^2 - 1)$ (Exercise 9.3). A proportional-derivative control law for this system has transfer function $C(s) = k_\mathrm{p} + k_\mathrm{d} s$ (see Table 9.1). Suppose that we choose $C(s) = \alpha(s-1)$. Compute the closed loop dynamics and show that the system has good tracking of reference signals but does not have good disturbance rejection properties.