Feedback Systems

An Introduction for Scientists and Engineers

SECOND EDITION

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Chapter 8

Output Feedback

One may separate the problem of physical realization into two stages: computation of the "best approximation" $\hat{x}(t_1)$ of the state from knowledge of $y(t)$ for $t \leq t_1$ and computation of $u(t_1)$ given $\hat{x}(t_1)$.

R. E. Kalman, "Contributions to the Theory of Optimal Control," 1960 [Kal60].

In this chapter we show how to use output feedback to modify the dynamics of the system through the use of observers. We introduce the concept of observability and show that if a system is observable, it is possible to recover the state from measurements of the inputs and outputs to the system. We then show how to design a controller with feedback from the observer state. A general controller with two degrees of freedom is obtained by adding feedforward. We illustrate by outlining a controller for a nonlinear system that also employs gain scheduling.

8.1 Observability

In Section 7.2 of the previous chapter it was shown that it is possible to find a state feedback law that gives desired closed loop eigenvalues provided that the system is reachable and that all the states are measured by sensors. For many situations, it is highly unrealistic to assume that all the states are measured. In this section we investigate how the state can be estimated by using a mathematical model and a few measurements. It will be shown that computation of the states can be carried out by a dynamical system called an *observer*.

Definition of Observability

Consider a system described by a set of differential equations

$$\frac{dx}{dt} = Ax + Bu, \qquad y = Cx + Du, \tag{8.1}$$

where $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}^p$ the input, and $y \in \mathbb{R}^q$ the measured output. We wish to estimate the state of the system from its inputs and outputs, as illustrated

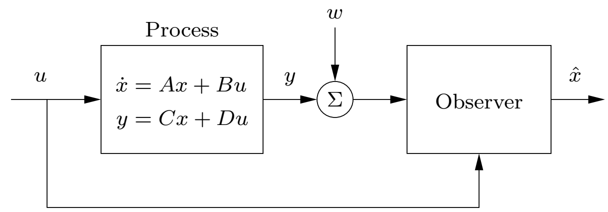


Figure 8.1: Block diagram for an observer. The observer uses the process measurement y (possibly corrupted by noise w) and the input u to estimate the current state of the process, denoted $\hat{x}$.

in Figure 8.1. In some situations we will assume that there is only one measured signal, i.e., that the signal y is a scalar and that C is a (row) vector. This signal may be corrupted by noise w, although we shall start by considering the noise-free case. We write $\hat{x}$ for the state estimate given by the observer.

Definition 8.1 (Observability)**.** A linear system is *observable* if for every $T > 0$ it is possible to determine the state of the system $x(T)$ through measurements of $y(t)$ and $u(t)$ on the interval $[0, T]$.

The definition above holds for nonlinear systems as well, and the results discussed here have extensions to the nonlinear case.

The problem of observability is one that has many important applications, even outside feedback systems. If a system is observable, then there are no "hidden" dynamics inside it; we can understand everything that is going on through observation (over time) of the inputs and outputs. As we shall see, the problem of observability is of significant practical interest because it will determine if a set of sensors is sufficient for controlling a system. Sensors combined with a mathematical model of the system can also be viewed as a "virtual sensor" that gives information about variables that are not measured directly. The process of reconciling signals from many sensors using mathematical models is also called *sensor fusion*.

Testing for Observability

When discussing reachability in the previous chapter, we neglected the output and focused on the state. Similarly, it is convenient here to initially neglect the input and focus on the autonomous system

$$\frac{dx}{dt} = Ax, \qquad y = Cx, \tag{8.2}$$

where $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^q$. We wish to understand when it is possible to determine the state from observations of the output.

The output itself gives the projection of the state onto vectors that are rows of the matrix C. The observability problem can immediately be solved if $n = q$ (number of outputs equals number of states) and the matrix C is invertible. If the matrix is not square and invertible, we can take derivatives of the output to obtain

$$\frac{dy}{dt} = C\frac{dx}{dt} = CAx.$$

From the derivative of the output we thus get the projection of the state on vectors that are rows of the matrix CA. Proceeding in this way, we get at every time t

$$\begin{pmatrix} y(t) \\ \dot{y}(t) \\ \ddot{y}(t) \\ \vdots \\ y^{(n-1)}(t) \end{pmatrix} = \begin{pmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{n-1} \end{pmatrix} x(t). \tag{8.3}$$

We thus find that the state at time t can be determined from the output and its derivatives at time t if the *observability matrix*

$$W_{\mathrm{o}} = \begin{pmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{n-1} \end{pmatrix} \tag{8.4}$$

has full row rank (n independent rows). As in the case of reachability, it turns out that we need not consider any derivatives higher than $n-1$ (this is an application of the Cayley–Hamilton theorem [Exercise 7.3]).

The calculation can easily be extended to systems with inputs and many measured signals. The state is then given by a linear combination of inputs and outputs and their higher derivatives. The observability criterion is unchanged. We leave this case as an exercise for the reader.

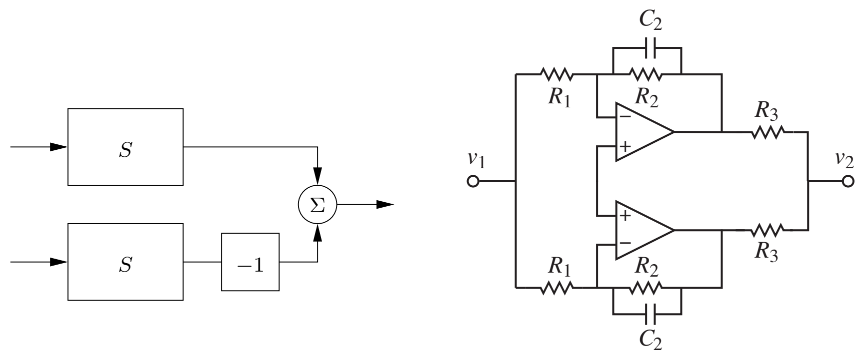
In practice, differentiation of the output can give large errors when there is measurement noise, and therefore the method sketched above is not particularly practical. We will address this issue in more detail in the next section, but for now we have the following basic result.

Theorem 8.1 (Observability rank condition)**.** *A linear system of the form* (8.1) *is observable if and only if the observability matrix W_o is full row rank.*

Proof. The sufficiency of the observability rank condition follows from the previous analysis. To prove necessity, suppose that the system is observable but W_{o} is not full row rank. Let $v \in \mathbb{R}^n$, $v \neq 0$, be a vector in the null space of W_{o}, so that $W_{\mathrm{o}} v = 0$. (Such a v exists using the fact that the row and column rank of a matrix are always equal.) If we let $x(0) = v$ be the initial condition for the system and choose $u = 0$, then the output is given by $y(t) = Ce^{At}v$. Since e^{At} can be written as a power series in A and since A^n and higher powers can be rewritten in terms of lower powers of A (by the Cayley–Hamilton theorem), it follows that $y(t)$ will be identically zero (the reader should fill in the missing steps). However, if both the input and output of the system are zero, then a valid estimate of the state is $\hat{x} = 0$ for all time, which is clearly incorrect since $x(0) = v \neq 0$. Hence by contradiction we must have that W_{o} is full row rank if the system is observable. □

Example 8.1 Compartment model

Consider the two-compartment model in Figure 4.18a, but assume that only the



(a) Block diagram (b) Op amp circuit

Figure 8.2: An unobservable system. Two identical subsystems have outputs that add together to form the overall system output. The individual states of the subsystem cannot be determined since the contributions of each to the output are not distinguishable. The circuit diagram on the right is an example of such a system.

concentration in the first compartment can be measured. The system is described by the linear system

$$\frac{dc}{dt} = \begin{pmatrix} -k_0 - k_1 & k_1 \\ k_2 & -k_2 \end{pmatrix} c + \begin{pmatrix} b_0 \\ 0 \end{pmatrix} u, \qquad y = \begin{pmatrix} 1 & 0 \end{pmatrix} c.$$

The first compartment represents the drug concentration in the blood plasma, and the second compartment the drug concentration in the tissue where it is active. To determine if it is possible to find the concentration in the tissue compartment from a measurement of blood plasma, we investigate the observability of the system by forming the observability matrix

$$W_{\mathrm{o}} = \begin{pmatrix} C \\ CA \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -k_0 - k_1 & k_1 \end{pmatrix}.$$

The rows are linearly independent if $k_1 \neq 0$, and under this condition it is thus possible to determine the concentration of the drug in the active compartment from measurements of the drug concentration in the blood. ∇

It is useful to have an understanding of the mechanisms that make a system unobservable. Such a system is shown in Figure 8.2. The system is composed of two identical systems whose outputs are subtracted. It seems intuitively clear that it is not possible to deduce the states from the output since we cannot deduce the individual output contributions from the difference. This can also be seen formally (Exercise 8.2).

Observable Canonical Form

As in the case of reachability, certain canonical forms will be useful in studying observability. A linear single-input, single-output state space system is in *observable*

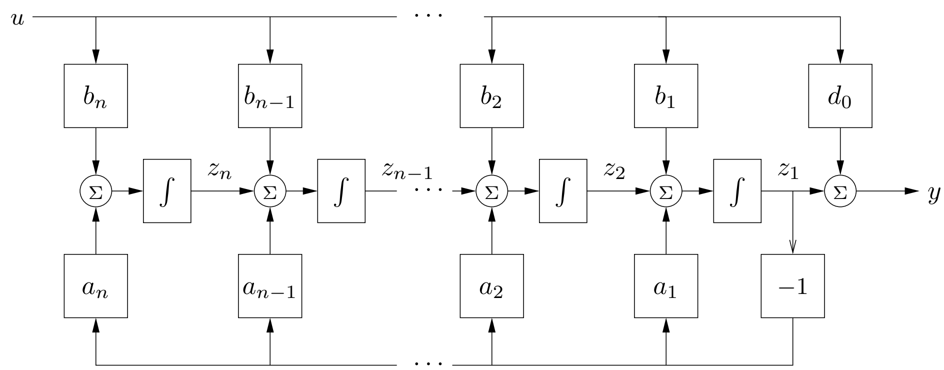


Figure 8.3: Block diagram of a system in observable canonical form. The states of the system are represented by individual integrators whose inputs are a weighted combination of the next integrator in the chain, the first state (rightmost integrator), and the system input. The output is a combination of the first state and the input. Compare with the block diagram of the system in reachable form in Figure 7.4.

canonical form if its dynamics are given by

$$\frac{dz}{dt} = \begin{pmatrix} -a_1 & 1 & & 0 \\ -a_2 & 0 & \ddots & \\ \vdots & & \ddots & 1 \\ -a_n & & 0 & 0 \end{pmatrix} z + \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix} u,$$
$$y = \begin{pmatrix} 1 & 0 & \cdots & 0 \end{pmatrix} z + d_0\, u.$$

This definition can be extended to systems with many inputs; the only difference is that the vector multiplying u is replaced by a matrix.

Figure 8.3 is a block diagram for a system in observable canonical form. As in the case of reachable canonical form, we see that the coefficients in the system description appear directly in the block diagram. The characteristic polynomial for a system in observable canonical form is

$$\lambda(s) = s^n + a_1 s^{n-1} + \cdots + a_{n-1} s + a_n. \tag{8.5}$$

It is possible to reason about the observability of a system in observable canonical form by studying the block diagram. If the input u and the output y are available, the state z_1 can clearly be computed. Differentiating z_1, we obtain the input to the integrator that generates z_1, and we can now obtain $z_2 = \dot{z}_1 + a_1 z_1 - b_1 u$. Proceeding in this way, we can compute all states. The computation will, however, require that the signals be differentiated.

To check observability more formally, we compute the observability matrix for a system in observable canonical form, which is given by

$$\tilde{W}_o = \begin{pmatrix} 1 & & & 0 \\ -a_1 & 1 & & \\ -a_1^2 - a_2 & -a_1 & 1 & \\ \vdots & \vdots & & \ddots \\ * & * & \cdots & 1 \end{pmatrix},$$

where * represents an entry whose exact value is not important. The columns of this matrix are linearly independent (since it is lower triangular), and hence W_o is invertible. A straightforward but tedious calculation shows that the inverse of the observability matrix has a simple form given by

$$\tilde{W}_o^{-1} = \begin{pmatrix} 1 & & & & 0 \\ a_1 & 1 & & & \\ a_2 & a_1 & 1 & & \\ \vdots & \vdots & & \ddots & \\ & & & 1 & \\ a_{n-1} & a_{n-2} & \cdots & a_1 & 1 \end{pmatrix}.$$

As in the case of reachability, it turns out that a system is observable if and only if there exists a transformation T that converts the system into observable canonical form. This is useful for proofs since it lets us assume that a system is in observable canonical form without any loss of generality. The observable canonical form may be poorly conditioned numerically.

8.2 State Estimation

Having defined the concept of observability, we now return to the question of how to construct an observer for a system. We will look for observers that can be represented as a linear dynamical system that takes the inputs and outputs of the system we are observing and produces an estimate of the system's state. That is, we wish to construct a dynamical system of the form

$$\frac{d\hat{x}}{dt} = F\hat{x} + Gu + Hy,$$

where u and y are the input and output of the original system and $\hat{x} \in \mathbb{R}^n$ is an estimate of the state with the property that $\hat{x}(t) \to x(t)$ as $t \to \infty$.

The Observer

We consider the system in equation (8.1) with D set to zero to simplify the exposition:

$$\frac{dx}{dt} = Ax + Bu, \qquad y = Cx. \tag{8.6}$$

We can attempt to determine the state simply by simulating the equations with the correct input. An estimate of the state is then given by

$$\frac{d\hat{x}}{dt} = A\hat{x} + Bu. \tag{8.7}$$

To find the properties of this estimate, introduce the estimation error $\tilde{x} = x - \hat{x}$. It follows from equations (8.6) and (8.7) that

$$\frac{d\tilde{x}}{dt} = A\tilde{x}.$$

If the dynamics matrix A has all its eigenvalues in the left half-plane, the error $\tilde{x}$ will go to zero, and hence equation (8.7) is a dynamical system whose output converges to the state of the system (8.6). However, the convergence might be slower than desired.

The observer given by equation (8.7) uses only the process input u; the measured signal does not appear in the equation. We must also require that the system be stable, and essentially our estimator converges because the transient dynamics of both the observer and the estimator are going to zero. This is not very useful in a control design context since we want to have our estimate converge quickly to a nonzero state so that we can make use of it in our controller. We will therefore attempt to modify the observer so that the output is used and its convergence properties can be designed to be fast relative to the system's dynamics. This version will also work for unstable systems.

Consider the observer

$$\frac{d\hat{x}}{dt} = A\hat{x} + Bu + L(y - C\hat{x}). \tag{8.8}$$

This can be considered as a generalization of equation (8.7). Feedback from the measured output is provided by adding the term $L(y - C\hat{x})$, which is proportional to the difference between the observed output and the output predicted by the observer. It follows from equations (8.6) and (8.8) that

$$\frac{d\tilde{x}}{dt} = (A - LC)\tilde{x}.$$

If the matrix L can be chosen in such a way that the matrix $A - LC$ has eigenvalues with negative real parts, the error $\tilde{x}$ will go to zero. The convergence rate is determined by an appropriate selection of the eigenvalues.

Notice the similarity between the problems of finding a state feedback and finding the observer. State feedback design by eigenvalue assignment is equivalent to finding a matrix K so that $A - BK$ has given eigenvalues. Designing an observer with prescribed eigenvalues is equivalent to finding a matrix L so that $A - LC$ has given eigenvalues. Since the eigenvalues of a matrix and its transpose are the same we can establish the following equivalences:

$$A \leftrightarrow A^T, \qquad B \leftrightarrow C^T, \qquad K \leftrightarrow L^T, \qquad W_\mathrm{r} \leftrightarrow W_\mathrm{o}^T. \tag{8.9}$$

The observer design problem is the *dual* of the state feedback design problem. Using the results of Theorem 7.3, we get the following theorem on observer design.

Theorem 8.2 (Observer design by eigenvalue assignment)**.** *Consider the system given by*

$$\frac{dx}{dt} = Ax + Bu, \qquad y = Cx, \tag{8.10}$$

with one input and one output. Let $\lambda(s) = s^n + a_1 s^{n-1} + \cdots + a_{n-1}s + a_n$ be the characteristic polynomial for A. If the system is observable, then the dynamical system

$$\frac{d\hat{x}}{dt} = A\hat{x} + Bu + L(y - C\hat{x}) \tag{8.11}$$

is an observer for the system, with L chosen as

$$L = W_o^{-1}\widetilde{W}_o \begin{pmatrix} p_1 - a_1 \\ p_2 - a_2 \\ \vdots \\ p_n - a_n \end{pmatrix} \tag{8.12}$$

and the matrices W_o and $\widetilde{W}_o$ given by

$$W_o = \begin{pmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{pmatrix}, \qquad \tilde{W}_o = \begin{pmatrix} 1 & & & & 0 \\ a_1 & 1 & & & \\ a_2 & a_1 & 1 & & \\ & & & \ddots & \\ \vdots & \vdots & & & \\ & & & 1 & \\ a_{n-1} & a_{n-2} & \cdots & a_1 & 1 \end{pmatrix}^{-1}.$$

The resulting observer error $\tilde{x} = x - \hat{x}$ is governed by a differential equation having the characteristic polynomial

$$p(s) = s^n + p_1 s^{n-1} + \cdots + p_n.$$

The dynamical system (8.11) is called an *observer* for (the states of) the system (8.10) because it will generate an approximation of the states of the system from its inputs and outputs. This form of an observer is a much more useful form than the one given by pure differentiation in equation (8.3).

Example 8.2 Compartment model

Consider the compartment model in Example 8.1, which is characterized by the matrices

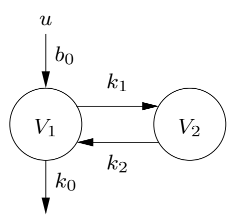
$$A = \begin{pmatrix} -k_0 - k_1 & k_1 \\ k_2 & -k_2 \end{pmatrix}, \qquad B = \begin{pmatrix} b_0 \\ 0 \end{pmatrix}, \qquad C = \begin{pmatrix} 1 & 0 \end{pmatrix}.$$

The observability matrix was computed in Example 8.1, where we concluded that the system was observable if $k_1 \neq 0$. The dynamics matrix has the characteristic polynomial

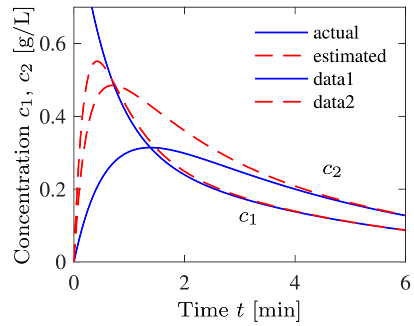
$$\lambda(s) = \det \begin{pmatrix} s + k_0 + k_1 & -k_1 \\ -k_2 & s + k_2 \end{pmatrix} = s^2 + (k_0 + k_1 + k_2)s + k_0 k_2.$$

Letting the desired characteristic polynomial of the observer be $s^2 + p_1 s + p_2$, equation (8.12) gives the observer gain

$$\begin{aligned} L &= \begin{pmatrix} 1 & 0 \\ -k_0 - k_1 & k_1 \end{pmatrix}^{-1} \begin{pmatrix} 1 & 0 \\ k_0 + k_1 + k_2 & 1 \end{pmatrix}^{-1} \begin{pmatrix} p_1 - k_0 - k_1 - k_2 \\ p_2 - k_0 k_2 \end{pmatrix} \\ &= \begin{pmatrix} p_1 - k_0 - k_1 - k_2 \\ (p_2 - p_1 k_2 + k_1 k_2 + k_2^2)/k_1 \end{pmatrix}. \end{aligned}$$



(a) Two compartment model



(b) Observer response

Figure 8.4: Observer for a two compartment system. A two compartment model is shown on the left. The observer measures the input concentration u and output concentration $y = c_1$ to determine the compartment concentrations, shown on the right. The true concentrations are shown by solid lines and the estimates generated by the observer by dashed lines.

Notice that the observability condition $k_1 \neq 0$ is essential. The behavior of the observer is illustrated by the simulation in Figure 8.4b. Notice how the observed concentrations approach the true concentrations. ∇

The observer is a dynamical system whose inputs are the process input u and the process output y. The rate of change of the estimate is composed of two terms. One term, $A\hat{x} + Bu$, is the rate of change computed from the model with $\hat{x}$ substituted for x. The other term, $L(y - \hat{y})$, is proportional to the difference $e = y - \hat{y}$ between measured output y and its estimate $\hat{y} = C\hat{x}$. The observer gain L is a matrix that determines how the error e is weighted and distributed among the state estimates. The observer thus combines measurements with a dynamical model of the system. A block diagram of the observer is shown in Figure 8.5.

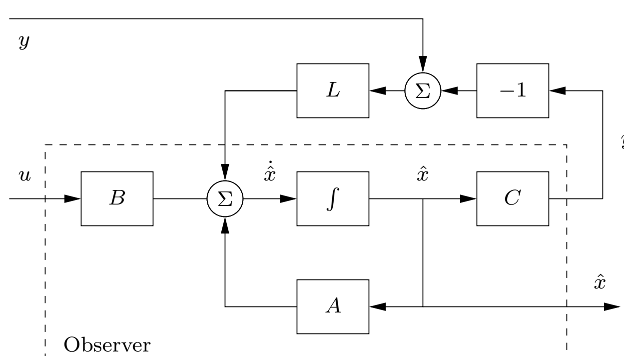


Figure 8.5: Block diagram of an observer. The observer takes the signals y and u as inputs and produces an estimate x. Notice that the observer contains a copy of the process model that is driven by $y - \hat{y}$ through the observer gain L.

Computing the Observer Gain

For simple low-order problems it is convenient to introduce the elements of the observer gain L as unknown parameters and solve for the values required to give the desired characteristic polynomial, as illustrated in the following example.

Example 8.3 Vehicle steering
The normalized linear model for vehicle steering derived in Examples 6.13 and 7.4 gives the following state space model dynamics relating lateral path deviation y to steering angle u:

$$\frac{dx}{dt} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} x + \begin{pmatrix} \gamma \\ 1 \end{pmatrix} u, \qquad y = \begin{pmatrix} 1 & 0 \end{pmatrix} x. \tag{8.13}$$

Recall that the state x_1 represents the lateral path deviation and that x_2 represents the turning rate. We will now derive an observer that uses the system model to determine the turning rate from the measured path deviation.

The observability matrix is

$$W_\mathrm{o} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

i.e., the identity matrix. The system is thus observable, and the eigenvalue assignment problem can be solved. We have

$$A - LC = \begin{pmatrix} -l_1 & 1 \\ -l_2 & 0 \end{pmatrix},$$

which has the characteristic polynomial

$$\det\,(sI - A + LC) = \det \begin{pmatrix} s + l_1 & -1 \\ l_2 & s \end{pmatrix} = s^2 + l_1 s + l_2.$$

Assuming that we want to have an observer with the characteristic polynomial

$$s^2 + p_1 s + p_2 = s^2 + 2\zeta_\mathrm{o}\omega_\mathrm{o} s + \omega_\mathrm{o}^2,$$

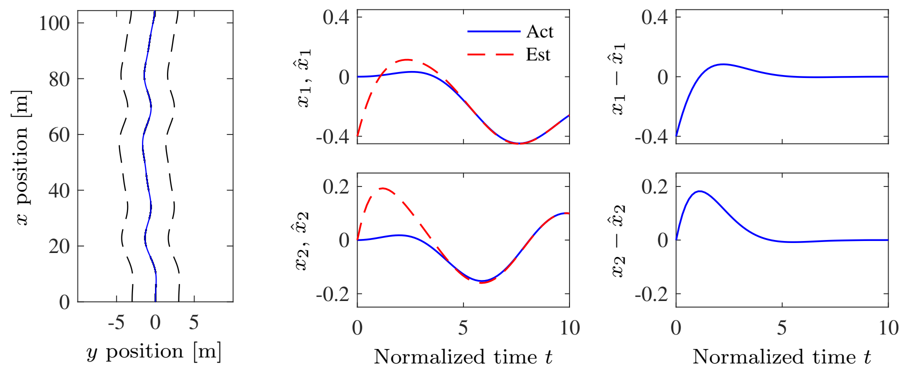
the observer gains should be chosen as

$$l_1 = p_1 = 2\zeta_\mathrm{o}\omega_\mathrm{o}, \qquad l_2 = p_2 = \omega_\mathrm{o}^2.$$

The observer is then

$$\frac{d\hat{x}}{dt} = A\hat{x} + Bu + L(y - C\hat{x}) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \hat{x} + \begin{pmatrix} \gamma \\ 1 \end{pmatrix} u + \begin{pmatrix} l_1 \\ l_2 \end{pmatrix} (y - \hat{x}_1).$$

A simulation of the observer for a vehicle driving on a curvy road is shown in Figure 8.6. Figure 8.6a shows the trajectory of the vehicle on the road, as viewed from above. The response of the observer is shown in Figure 8.6b, where time is normalized to the vehicle length. We see that the observer error settles in about 4 vehicle lengths. ∇



(a) Vehicle trajectory (b) State trajectories, estimates, and errors

Figure 8.6: Simulation of an observer for a vehicle driving on a curvy road. (a) The vehicle trajectory, as viewed from above, with the lane boundaries shown as dashed lines. (b) The response of the observer with an initial position error. The plots on the left show the lateral deviation x_1 and the lateral velocity x_2 with solid lines and their estimates $\hat{x}_1$ and $\hat{x}_2$ with dashed lines. The plots on the right show the estimation errors. The parameters used to design the estimator were $\omega_o = 1$ and $\zeta_o = 0.7$.

To compute the observer gains for systems of high order we have to use numerical calculations. The duality between the design of a state feedback and the design of an observer means that the computer algorithms for state feedback can also be used for the observer design; we simply use the transpose of the dynamics matrix and the output matrix. The MATLAB command `acker`, which essentially is a direct implementation of the calculations given in Theorem 8.2, can be used for systems with one output. The MATLAB command `place` can be used for systems with many outputs. It is also better conditioned numerically.

Requirements on a control system typically involve fast response to reference inputs and disturbances at the same time as avoiding amplification of noise. Choosing a fast observer gives fast convergence but the observer gains will be high and the estimated state will be sensitive to measurement noise. If noise characteristics are known it is possible to find the best compromise, as will be discussed in Section 8.4, the observer is then called a *Kalman filter*.

8.3 Control Using Estimated State

In this section we will consider a state space system of the form

$$\frac{dx}{dt} = Ax + Bu, \qquad y = Cx. \tag{8.14}$$

We wish to design a feedback controller for the system where only the output is measured. Notice that we have assumed that there is no direct term in the system ($D = 0$), which is often a realistic assumption. The presence of a direct term in combination with a controller having proportional action creates an algebraic loop, which will be discussed in Section 9.4. The problem can still be solved even if there is a direct term, but the calculations are more complicated.

As before, we will assume that u and y are scalars. We also assume that the system is reachable and observable. In Chapter 7 we found a feedback of the form

$$u = -Kx + k_\mathrm{f} r$$

for the case that all states could be measured, and in Section 8.2 we developed an observer that can generate estimates of the state $\hat{x}$ based on inputs and outputs. In this section we will combine the ideas of these sections to find a feedback that gives desired closed loop eigenvalues for systems where only outputs are available for feedback.

If all states are not measurable, it seems reasonable to try the feedback

$$u = -K\hat{x} + k_\mathrm{f} r, \tag{8.15}$$

where $\hat{x}$ is the output of an observer of the state, i.e.,

$$\frac{d\hat{x}}{dt} = A\hat{x} + Bu + L(y - C\hat{x}). \tag{8.16}$$

It is not clear that such a combination will have the desired effect. To explore this, note that since the system (8.14) and the observer (8.16) are both of state dimension n, the closed loop system has state dimension $2n$ with state $(x, \hat{x})$. The evolution of the states is described by equations (8.14)–(8.16). To analyze the closed loop system, we change coordinates and replace the estimated state variable $\hat{x}$ by the estimation error

$$\tilde{x} = x - \hat{x}. \tag{8.17}$$

Subtraction of equation (8.16) from equation (8.14) gives

$$\frac{d\tilde{x}}{dt} = Ax - A\hat{x} - L(Cx - C\hat{x}) = A\tilde{x} - LC\tilde{x} = (A - LC)\tilde{x}.$$

Returning to the process dynamics, introducing u from equation (8.15) into equation (8.14) and using equation (8.17) to eliminate $\hat{x}$ gives

$$\begin{aligned}\frac{dx}{dt} &= Ax + Bu = Ax - BK\hat{x} + Bk_\mathrm{f} r = Ax - BK(x - \tilde{x}) + Bk_\mathrm{f} r\\ &= (A - BK)x + BK\tilde{x} + Bk_\mathrm{f} r.\end{aligned}$$

The closed loop system is thus governed by

$$\frac{d}{dt}\begin{pmatrix} x \\ \tilde{x} \end{pmatrix} = \begin{pmatrix} A - BK & BK \\ 0 & A - LC \end{pmatrix}\begin{pmatrix} x \\ \tilde{x} \end{pmatrix} + \begin{pmatrix} Bk_\mathrm{f} \\ 0 \end{pmatrix} r. \tag{8.18}$$

Notice that the state $\tilde{x}$, representing the observer error, is not affected by the reference signal r. This is desirable since we do not want the reference signal to generate observer errors.

Since the dynamics matrix is block diagonal, we find that the characteristic polynomial of the closed loop system is

$$\lambda(s) = \det(sI - A + BK)\det(sI - A + LC).$$

This polynomial is a product of two terms: the characteristic polynomial of the closed loop system obtained with state feedback $\det(sI - A + BK)$ and the characteristic polynomial of the observer $\det(sI - A + LC)$. The design procedure thus separates into two subproblems: design of a state feedback and design of an observer. The feedback (8.15) that was motivated heuristically therefore provides an elegant solution to the eigenvalue assignment problem for output feedback. The result is summarized as follows.

Theorem 8.3 (Eigenvalue assignment by output feedback)**.** *Consider the system*

$$\frac{dx}{dt} = Ax + Bu, \qquad y = Cx.$$

The controller described by

$$\begin{aligned}\frac{d\hat{x}}{dt} &= A\hat{x} + Bu + L(y - C\hat{x}) = (A - BK - LC)\hat{x} + Bk_f r + Ly,\\ u &= -K\hat{x} + k_f r\end{aligned}$$

gives a closed loop system with the characteristic polynomial

$$\lambda(s) = \det(sI - A + BK)\det(sI - A + LC).$$

This polynomial can be assigned arbitrary roots if the system is reachable and observable.

The controller has a strong intuitive appeal: it can be thought of as being composed of two parts: state feedback and an observer. The controller is now a dynamic compensator with internal state dynamics generated by the observer. The control action makes use of feedback from the estimated states $\hat{x}$. The feedback gain K can be computed as if all state variables can be measured, and it depends only on A and B. The observer gain L depends only on A and C. The property that the eigenvalue assignment for output feedback can be separated into an eigenvalue assignment for a state feedback and an observer is called the *separation principle*.

A block diagram of the controller is shown in Figure 8.7. Notice that the controller contains a dynamical model of the plant. This is called the *internal model principle*: the controller contains a model of the process being controlled.

Design of control systems involves a balance between achieving high performance while maintaining adequate robustness in the presence of uncertainties. It is not obvious how such properties are reflected in the closed loop eigenvalues. It is therefore important to evaluate the design for example by plotting time responses to get more insight into the properties of the design. Additional discussion is presented in Section 14.5, where we consider the robustness of eigenvalue assignment (pole placement) design and also give some design rules.

Example 8.4 Vehicle steering
Consider again the normalized linear model for vehicle steering in Example 7.4. The dynamics relating the steering angle u to the lateral path deviation y are given by the state space model (8.13). Combining the state feedback derived in Example 7.4

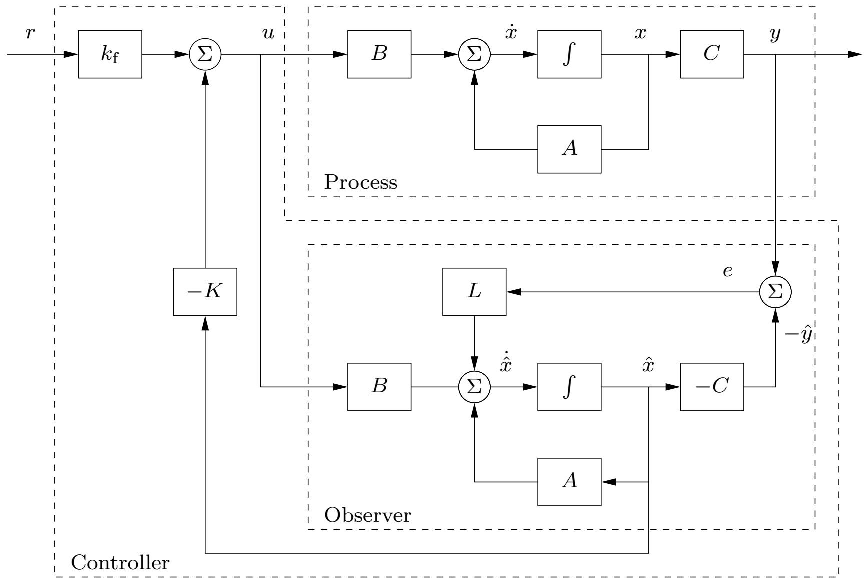


Figure 8.7: Block diagram of an observer-based control system. The observer uses the measured output y and the input u to construct an estimate of the state. This estimate is used by a state feedback controller to generate the corrective input. The controller consists of the observer and the state feedback; the observer is identical to that in Figure 8.5.

with the observer determined in Example 8.3, we find that the controller is given by

$$\frac{d\hat{x}}{dt} = A\hat{x} + Bu + L(y - C\hat{x}) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \hat{x} + \begin{pmatrix} \gamma \\ 1 \end{pmatrix} u + \begin{pmatrix} l_1 \\ l_2 \end{pmatrix} (y - \hat{x}_1),$$
$$u = -K\hat{x} + k_\mathrm{f} r = k_1(r - \hat{x}_1) - k_2\hat{x}_2.$$

Elimination of the variable u gives

$$\begin{aligned}\frac{d\hat{x}}{dt} &= (A - BK - LC)\hat{x} + Ly + Bk_\mathrm{f} r \\ &= \begin{pmatrix} -l_1 - \gamma k_1 & 1 - \gamma k_2 \\ -k_1 - l_2 & -k_2 \end{pmatrix} \hat{x} + \begin{pmatrix} l_1 \\ l_2 \end{pmatrix} y + \begin{pmatrix} \gamma \\ 1 \end{pmatrix} k_1 r,\end{aligned}$$

where we have set $k_\mathrm{f} = k_1$ as described in Example 7.4. The controller is a dynamical system of second order, with two inputs y and r and one output u. Figure 8.8 shows a simulation of the system when the vehicle is driven along a curvy road. Since we are using a normalized model, the length unit is the vehicle length and the time unit is the time it takes to travel one vehicle length. The estimator is initialized with all states equal to zero but the real system has an initial lateral position of 0.8. The figures show that the estimates converge quickly to their true values. The vehicle roughly tracks the desired path, but there are errors because the road is curving. The tracking error can be improved by introducing feedforward (Section 8.5). ∇

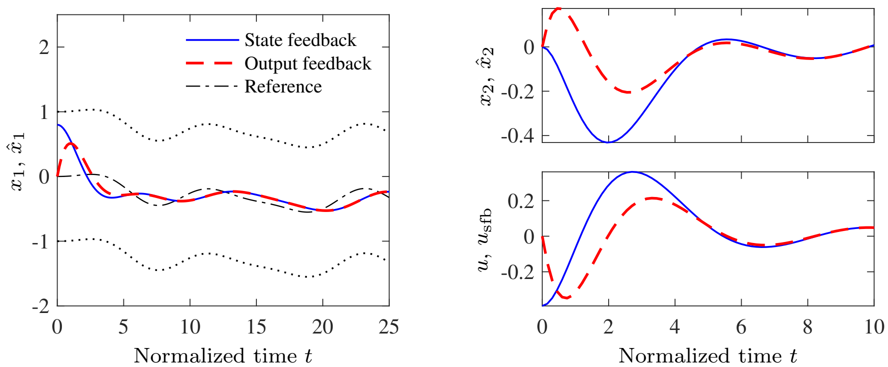


Figure 8.8: Simulation of a vehicle driving on a curvy road with a controller based on state feedback and an observer. The left plot shows the lane boundaries (dotted), the vehicle position (solid), and its estimate (dashed), the upper right plot shows the velocity (solid) and its estimate (dashed), and the lower right plot shows the control signal using state feedback (solid) and the control signal using the estimated state (dashed).

Kalman's Decomposition of a Linear System

In this chapter and the previous one we have seen that two fundamental properties of a linear input/output system are reachability and observability. It turns out that these two properties can be used to classify the dynamics of a system. The key result is Kalman's decomposition theorem, which says that a linear system can be divided into four subsystems: $\Sigma_{\rm ro}$ which is reachable and observable, $\Sigma_{\rm r\bar{o}}$ which is reachable but not observable, $\Sigma_{\rm \bar{r}o}$ which is not reachable but is observable, and $\Sigma_{\rm \overline{ro}}$ which is neither reachable nor observable.

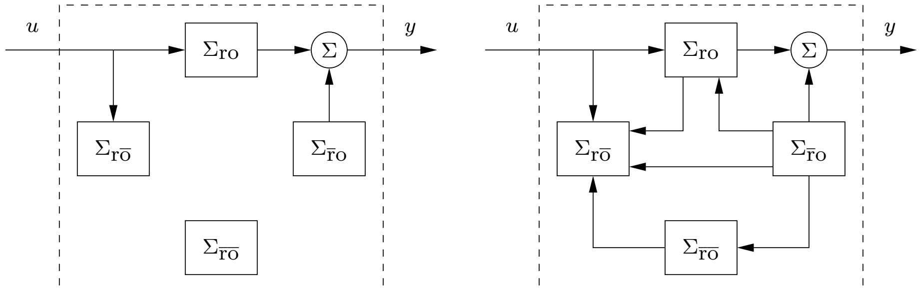
We will first consider this in the special case of systems with one input and one output, and where the matrix A has distinct eigenvalues. In this case we can find a set of coordinates such that the A matrix is diagonal and, with some additional reordering of the states, the system can be written as

$$\begin{aligned} \frac{dx}{dt} &= \begin{pmatrix} A_{\rm ro} & 0 & 0 & 0 \\ 0 & A_{\rm r\bar{o}} & 0 & 0 \\ 0 & 0 & A_{\rm \bar{r}o} & 0 \\ 0 & 0 & 0 & A_{\rm \overline{ro}} \end{pmatrix} x + \begin{pmatrix} B_{\rm ro} \\ B_{\rm r\bar{o}} \\ 0 \\ 0 \end{pmatrix} u, \\ y &= \begin{pmatrix} C_{\rm ro} & 0 & C_{\rm \bar{r}o} & 0 \end{pmatrix} x + Du. \end{aligned} \tag{8.19}$$

All states x_k such that $B_k \neq 0$ are reachable, and all states such that $C_k \neq 0$ are observable. If we set the initial state to zero (or equivalently look at the steady-state response if A is stable), the states given by $x_{\bar{r}o}$ and $x_{\overline{\rm ro}}$ will be zero and $x_{r\bar{o}}$ does not affect the output. Hence the output y can be determined from the system

$$\frac{dx_{\rm ro}}{dt} = A_{\rm ro}x_{\rm ro} + B_{\rm ro}u, \qquad y = C_{\rm ro}x_{\rm ro} + Du.$$

Thus from the input/output point of view, it is only the reachable and observable dynamics that matter. A block diagram of the system illustrating this property is given in Figure 8.9a.



(a) Distinct eigenvalues (b) General case

Figure 8.9: Kalman's decomposition of a linear system. The decomposition in (a) is for a system with distinct eigenvalues and the one in (b) is the general case. The system is broken into four subsystems, representing the various combinations of reachable and observable states. The input/output relationship only depends on the subset of states that are both reachable and observable.

The general case of the Kalman decomposition is more complicated and requires some additional linear algebra; see the original paper by Kalman, Ho, and Narendra [KHN63]. The key result is that the state space can still be decomposed into four parts, but there will be additional coupling so that the equations have the form

$$\begin{aligned}\frac{dx}{dt} &= \begin{pmatrix} A_{\mathrm{ro}} & 0 & * & 0 \\ * & A_{\mathrm{r\bar{o}}} & * & * \\ 0 & 0 & A_{\mathrm{\bar{r}o}} & 0 \\ 0 & 0 & * & A_{\mathrm{\bar{r}\bar{o}}} \end{pmatrix} x + \begin{pmatrix} B_{\mathrm{ro}} \\ B_{\mathrm{r\bar{o}}} \\ 0 \\ 0 \end{pmatrix} u, \\ y &= \begin{pmatrix} C_{\mathrm{ro}} & 0 & C_{\mathrm{\bar{r}o}} & 0 \end{pmatrix} x,\end{aligned} \tag{8.20}$$

where $*$ denotes block matrices of appropriate dimensions. If $x_{\mathrm{\bar{r}o}}(0) = 0$ then the input/output response of the system is given by

$$\frac{dx_{\mathrm{ro}}}{dt} = A_{\mathrm{ro}}x_{\mathrm{ro}} + B_{\mathrm{ro}}u, \qquad y = C_{\mathrm{ro}}x_{\mathrm{ro}} + Du, \tag{8.21}$$

which are the dynamics of the reachable and observable subsystem Σ_{ro}. A block diagram of the system is shown in Figure 8.9b.

The following example illustrates Kalman's decomposition.

Example 8.5 System and controller with feedback from observer states
Consider the system

$$\frac{dx}{dt} = Ax + Bu, \qquad y = Cx.$$

The following controller, based on feedback from the observer state, was given in Theorem 8.3:

$$\frac{d\hat{x}}{dt} = A\hat{x} + Bu + L(y - C\hat{x}), \qquad u = -K\hat{x} + k_{\mathrm{f}}r.$$

Introducing the states x and $\tilde{x} = x - \hat{x}$, the closed loop system can be written as

$$\frac{d}{dt}\begin{pmatrix} x \\ \tilde{x} \end{pmatrix} = \begin{pmatrix} A - BK & BK \\ 0 & A - LC \end{pmatrix}\begin{pmatrix} x \\ \tilde{x} \end{pmatrix} + \begin{pmatrix} Bk_{\mathrm{f}} \\ 0 \end{pmatrix} r, \qquad y = \begin{pmatrix} C & 0 \end{pmatrix}\begin{pmatrix} x \\ \tilde{x} \end{pmatrix},$$

which is a Kalman decomposition like the one shown in Figure 8.9b with only two subsystems Σ_{ro} and $\Sigma_{\bar{\mathrm{r}}\mathrm{o}}$. The subsystem Σ_{ro}, with state x, is reachable and observable, and the subsystem $\Sigma_{\bar{\mathrm{r}}\mathrm{o}}$, with state $\tilde{x}$, is not reachable but observable. It is natural that the state $\tilde{x}$ is not reachable from the reference signal r because it would not make sense to design a system where changes in the reference signal could generate observer errors. The relationship between the reference r and the output y is given by

$$\frac{dx}{dt} = (A - BK)x + Bk_{\mathrm{f}}r, \qquad y = Cx,$$

which is the same relationship as for a system with full state feedback. ∇

8.4 Kalman Filtering

One of the principal uses of observers in practice is to estimate the state of a system in the presence of *noisy* measurements. We have not yet treated noise in our analysis, and a full treatment of stochastic dynamical systems is beyond the scope of this text. In this section, we present a brief introduction to the use of stochastic systems analysis for constructing observers. We work primarily in discrete time to avoid some of the complications associated with continuous-time random processes and to keep the mathematical prerequisites to a minimum. This section assumes basic knowledge of random variables and stochastic processes; see Kumar and Varaiya [KV86] or Åström [Åst06] for the required material.

Discrete-Time Systems

Consider a discrete-time linear system with dynamics

$$x[k+1] = Ax[k] + Bu[k] + v[k], \qquad y[k] = Cx[k] + w[k], \tag{8.22}$$

where $v[k]$ and $w[k]$ are Gaussian white noise processes satisfying

$$\begin{aligned} &\mathbb{E}(v[k]) = 0, && \mathbb{E}(w[k]) = 0,\\ &\mathbb{E}(v[k]v^T[j]) = \begin{cases} 0 & \text{if } k \neq j,\\ R_v & \text{if } k = j, \end{cases} && \mathbb{E}(w[k]w^T[j]) = \begin{cases} 0 & \text{if } k \neq j,\\ R_w & \text{if } k = j, \end{cases}\\ &\mathbb{E}(v[k]w^T[j]) = 0. \end{aligned} \tag{8.23}$$

$\mathbb{E}(v[k])$ represents the expected value of $v[k]$ and $\mathbb{E}(v[k]v^T[j])$ is the covariance matrix. The matrices R_v and R_w are the covariance matrices for the process disturbance v and measurement noise w. (R_v is allowed to be singular if the disturbances do not affect all states.) We assume that the initial condition is also modeled as a Gaussian random variable with

$$\mathbb{E}(x[0]) = x_0, \qquad \mathbb{E}\left((x[0] - x_0)(x[0] - x_0)^T\right) = P_0. \tag{8.24}$$

We would like to find an estimate $\hat{x}[k]$ that minimizes the mean square error

$$P[k] = \mathbb{E}\left((x[k] - \hat{x}[k])(x[k] - \hat{x}[k])^T\right) \tag{8.25}$$

given the measurements $\{y(\kappa) : 0 \leq \kappa \leq k\}$. We consider an observer in the same basic form as derived previously:

$$\hat{x}[k+1] = A\hat{x}[k] + Bu[k] + L[k](y[k] - C\hat{x}[k]). \tag{8.26}$$

The following theorem summarizes the main result.

Theorem 8.4 (Kalman, 1961)**.** *Consider a random process* $x[k]$ *with dynamics given by equation* (8.22) *and noise processes and initial conditions described by equations* (8.23) *and* (8.24)*. The observer gain* L *that minimizes the mean square error is given by*

$$L[k] = AP[k]C^T(R_w + CP[k]C^T)^{-1},$$

where

$$\begin{aligned} P[k+1] &= (A - LC)P[k](A - LC)^T + R_v + LR_wL^T, \\ P[0] &= \mathbb{E}\left((x[0] = x_0)(x[0] - x_0)^T\right). \end{aligned} \tag{8.27}$$

Before we prove this result, we reflect on its form and function. First, note that the Kalman filter has the form of a *recursive* filter: given the mean square error $P[k] = \mathbb{E}((x[k] - \hat{x}[k])(x[k] - \hat{x}[k])^T)$ at time k, we can compute how the estimate and error *change*. Thus we do not need to keep track of old values of the output. Furthermore, the Kalman filter gives the estimate $\hat{x}[k]$ *and* the error covariance $P[k]$, so we can see how reliable the estimate is. It can also be shown that the Kalman filter extracts the maximum possible information about output data. If we form the residual between the measured output and the estimated output,

$$e[k] = y[k] - C\hat{x}[k],$$

we can show that for the Kalman filter the covariance matrix is

$$R_e(j,k) = \mathbb{E}(e[j]e^T[k]) = W[k]\delta_{jk}, \qquad \delta_{jk} = \begin{cases} 1 & \text{if } j = k, \\ 0 & \text{if } j \neq k. \end{cases}$$

In other words, the error is a white noise process, so there is no remaining dynamic information content in the error.

The Kalman filter is extremely versatile and can be used even if the process, noise, or disturbances are time-varying. When the system is time-invariant and *if* $P[k]$ converges, then the observer gain is constant:

$$L = APC^T(R_w + CPC^T),$$

where P satisfies

$$P = APA^T + R_v - APC^T\left(R_w + CPC^T\right)^{-1}CPA^T.$$

We see that the optimal gain depends on both the process noise and the measurement noise, but in a nontrivial way. Like the use of LQR to choose state feedback gains, the Kalman filter permits a systematic derivation of the observer gains given a description of the noise processes. The solution for the constant gain case is solved by the `dlqe` command in MATLAB.

Proof of theorem. We wish to minimize the mean square of the error $\mathbb{E}((x[k]-\hat{x}[k])(x[k]-\hat{x}[k])^T)$. We will define this quantity as $P[k]$ and then show that it satisfies the recursion given in equation (8.27). By definition,

$$\begin{aligned}P[k+1] &= \mathbb{E}\left((x[k+1]-\hat{x}[k+1])(x[k+1]-\hat{x}[k+1])^T\right)\\ &= (A-LC)P[k](A-LC)^T + R_v + LR_wL^T\\ &= AP[k]A^T + R_v - AP[k]C^TL^T - LCP[k]A^T\\ &\qquad + L(R_w + CP[k]C^T)L^T.\end{aligned}$$

Letting $R_\epsilon = (R_w + CP[k]C^T)$, we have

$$\begin{aligned}P[k+1] &= AP[k]A^T + R_v - AP[k]C^TL^T - LCP[k]A^T + LR_\epsilon L^T\\ &= AP[k]A^T + R_v + \left(L - AP[k]C^TR_\epsilon^{-1}\right)R_\epsilon\left(L - AP[k]C^TR_\epsilon^{-1}\right)^T\\ &\qquad - AP[k]C^TR_\epsilon^{-1}CP^T[k]A^T.\end{aligned}$$

To minimize this expression, we choose $L = AP[k]C^TR_\epsilon^{-1}$, and the theorem is proved. □

Continuous-Time Systems

The Kalman filter can also be applied to continuous-time stochastic processes. The mathematical derivation of this result requires more sophisticated tools, but the final form of the estimator is relatively straightforward.

Consider a continuous stochastic system

$$\begin{aligned}\frac{dx}{dt} &= Ax + Bu + v, \qquad && \mathbb{E}(v(s)v^T(t)) = R_v\delta(t-s),\\ y &= Cx + w, && \mathbb{E}(w(s)w^T(t)) = R_w\delta(t-s),\end{aligned} \tag{8.28}$$

where $\delta(\tau)$ is the unit impulse function, and the initial value is Gaussian with mean x_0 and covariance $P_0 = \mathbb{E}((x(0)-x_0)(x(0)-x_0)^T)$. Assume that the disturbance v and noise w are zero mean and Gaussian (but not necessarily time-invariant):

$$\mathrm{pdf}(v) = \frac{1}{\sqrt[n]{2\pi}\sqrt{\det R_v}}e^{-\frac{1}{2}v^TR_v^{-1}v}, \quad \mathrm{pdf}(w) \;=\; \frac{1}{\sqrt[q]{2\pi}\sqrt{\det R_w}}e^{-\frac{1}{2}w^TR_w^{-1}w}. \tag{8.29}$$

The model (8.28) is very general. We can model the dynamics both of the process and disturbances, as illustrated by the following example.

Example 8.6 Modeling a noisy sinusoidal disturbance
Consider a process whose dynamics are described by

$$\frac{dx}{dt} = x + u + v, \qquad y = x + w.$$

The disturbance v is a noisy sinusoidal disturbance with frequency ω_0 and w is white measurement noise. We model the oscillatory load disturbance as $v = z_1$, where

$$\frac{d}{dt}\begin{pmatrix} z_1\\ z_2\end{pmatrix} = \begin{pmatrix} -0.01\omega_0 & \omega_0\\ -\omega_0 & -0.01\omega_0\end{pmatrix}\begin{pmatrix} z_1\\ z_2\end{pmatrix} + \begin{pmatrix} 0\\ \omega_0\end{pmatrix} e,$$

and e is zero mean white noise with covariance function $r\delta(t)$.

Augmenting the system state with the states of the noise model by introducing the new state $\xi = (x, z_1, z_2)$, we obtain the model

$$\frac{d\xi}{dt} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & -0.01\omega_0 & \omega_0 \\ 0 & -\omega_0 & -0.01\omega_0 \end{pmatrix} \xi + \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} u + v, \quad y = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} \xi + w,$$

where v is white Gaussian noise with zero mean and the covariance $R_v\delta(t)$ with

$$R_v = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \omega_0^2 r \end{pmatrix}.$$

The model is in the standard form given by equations (8.28) and (8.29). ∇

We will now return to the filtering problem. Specifically, we wish to find the estimate $\hat{x}(t)$ that minimizes the mean square error $P(t) = \mathbb{E}((x(t) - \hat{x}(t))(x(t) - \hat{x}(t))^T)$ given $\{y(\tau) : 0 \le \tau \le t\}$.

Theorem 8.5 (Kalman–Bucy, 1961)**.** *The optimal estimator has the form of a linear observer*

$$\frac{d\hat{x}}{dt} = A\hat{x} + Bu + L(y - C\hat{x}), \qquad \hat{x}(0) = \mathbb{E}(x(0)),$$

where $L = PC^T R_w^{-1}$ *and* $P = \mathbb{E}((x(t) - \hat{x}(t))(x(t) - \hat{x}(t))^T)$ *and satisfies*

$$\frac{dP}{dt} = AP + PA^T - PC^T R_w^{-1} CP + R_v, \quad P(0) = \mathbb{E}\left((x(0) - x_0)(x(0) - x_0)^T\right). \tag{8.30}$$

All matrices A, B, C, R_v, R_w, P, and L can be time varying. The essential condition is that the Riccati equation (8.30) has a unique positive solution.

As in the discrete case, when the system is time-invariant and if $P(t)$ converges, the observer gain $L = PC^T R_w^{-1}$ is constant and P is the solution to

$$AP + PA^T - PC^T R_w^{-1} CP + R_v = 0, \tag{8.31}$$

which is called the *algebraic Riccati equation*.

Notice that there are a strong similarities between the Riccati equations (8.30) and (8.31) for the Kalman filtering problem and the corresponding equations (7.31) and (7.33) for the linear quadratic regulator (LQR). We have the equivalences

$$A \leftrightarrow A^T, \quad B \leftrightarrow C^T, \quad K \leftrightarrow L^T, \quad P \leftrightarrow S, \quad Q_x \leftrightarrow R_v, \quad Q_u \leftrightarrow R_w, \tag{8.32}$$

which we can compare with equation (8.9). The MATLAB command `kalman` can be used to compute optimal filter gains.

Example 8.7 Vectored thrust aircraft

The dynamics for a vectored thrust aircraft were considered in Examples 3.12 and 7.9. We consider the (linearized) lateral dynamics of the system, consisting of the subsystems whose states are given by $z = (x, \theta, \dot{x}, \dot{\theta})$. The dynamics of the linearized system can be obtained from Example 7.9 by extracting only the relevant states and outputs, giving

$$A = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -g & -c/m & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 \\ 0 \\ 0 \\ r/J \end{pmatrix}, \qquad C = \begin{pmatrix} 0 & 0 & 0 & 1 \end{pmatrix},$$

where the linearized state $\xi = z - z_e$ represents the system state linearized around the equilibrium point z_e. To design a Kalman filter for the system, we must include a description of the process disturbances and the sensor noise. We thus augment the system to have the form

$$\frac{d\xi}{dt} = A\xi + Bu + Fv, \qquad y = C\xi + w,$$

where F represents the structure of the disturbances (including the effects of nonlinearities that we have ignored in the linearization), v represents the disturbance source (modeled as zero mean, Gaussian white noise), and w represents that measurement noise (also zero mean, Gaussian, and white).

For this example, we choose F as the identity matrix and choose disturbances v, $i = 1, \dots, n$, to be independent random variables with covariance matrix elements given by $R_{ii} = 0.1$, $R_{ij} = 0$, $i \neq j$. The sensor noise is a single random variable that we model as white noise having covariance $R_w = 10^{-4}$. Using the same parameters as before, the resulting Kalman gain is given by

$$L = PC^T R_w^{-1} = \begin{pmatrix} 37.0 \\ -46.9 \\ 185 \\ -31.6 \end{pmatrix} \quad \text{where} \quad AP + PA^T - PC^T R_w^{-1} CP + R_v = 0.$$

The performance of the estimator is shown in Figure 8.10a. We see that while the estimator roughly tracks the system state, it contains significant overshoot in the state estimate and has significant error in the estimate for θ even after 2 seconds, which can lead to poor performance in a closed loop setting.

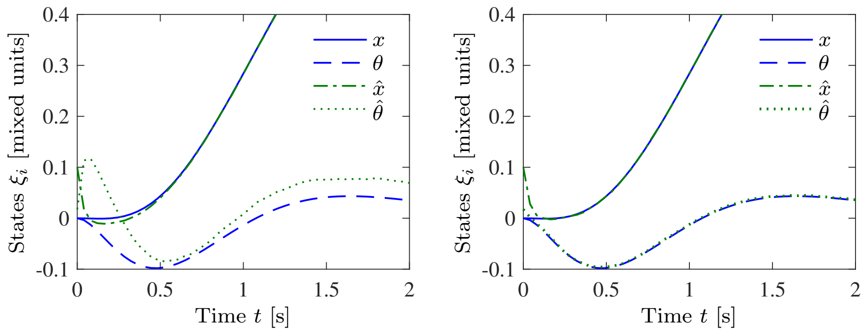
To improve the performance of the estimator, we explore the impact of adding a new output measurement. Suppose that instead of measuring just the output position x, we also measure the orientation of the aircraft θ. The output becomes

$$y = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \xi + \begin{pmatrix} w_1 \\ w_2 \end{pmatrix},$$

and if we assume that w_1 and w_2 are independent white noise sources each with covariance $R_{w_i} = 10^{-4}$, then the optimal estimator gain matrix becomes

$$L = \begin{pmatrix} 32.6 & -0.150 \\ -0.150 & 32.6 \\ 32.7 & -9.79 \\ -0.0033 & 31.6 \end{pmatrix}.$$

These gains provide good immunity to noise and high performance, as illustrated in Figure 8.10b. ∇



(a) Position measurement only

(b) Position and orientation

Figure 8.10: Kalman filter response for a (linearized) vectored thrust aircraft with disturbances and noise during the initial portion of a step response. In the first design (a) only the lateral position of the aircraft is measured. Adding a direct measurement of the roll angle produces a much better observer (b). The initial estimator state for both simulations is $(0.1, 0.0175, 0.01, 0)$ and the controller gains are $K = (-1, 7.9, -1.6, 2.1)$ and $k_f = -1$.

Linear Quadratic Gaussian Control (LQG)

In Section 7.5 we considered optimization of the criterion (7.29) when the control $u(t)$ could be a function of the state $x(t)$. We will now explore the same problem for the stochastic system (8.28) where the control $u(t)$ is a function of the output $y(t)$.

Consider the system given by equation (8.28) where the initial state is Gaussian with mean x_0 and covariance P_0 and the disturbances v and w are characterized by equation (8.29). Assume that the requirement can be captured by the cost function

$$J = \min_u \mathbb{E}\left(\int_0^{t_f} (x^T Q_x x + u^T Q_u u)\, dt + x^T(t_f) Q_f x(t_f) \right), \tag{8.33}$$

where we minimize over all controls such that $u(t)$ is a function of all measurements $y(\tau)$, $0 \le \tau \le t$ obtained up to time t.

The optimal control law is simply $u(t) = -K\hat{x}(t)$ where $K = SBQ_u^{-1}$ and S is the solution of the Riccati equation (7.31) (for the linear quadratic regulator) and $\hat{x}(t)$ is given by the Kalman filter (Theorem 8.5). The solution of the problem can thus be separated into a deterministic control problem (LQR) and an optimal filtering problem. This remarkable result is also known as the *separation principle*, as mentioned briefly in Section 8.3.

The minimum cost function is

$$\min J = x_0^T S(0) x_0 + \mathrm{Tr}\,(S(0)P_0) + \int_0^{t_f} \mathrm{Tr}\,(R_v S)\, dt + \int_0^{t_f} \mathrm{Tr}\,(L^T Q_u L P)\, dt,$$

where Tr is the trace of a matrix. The first two terms represent the cost of the mean x_0 and covariance P_0 of the initial state, the third term represents the cost due to the load disturbance, and the last term represents the cost of prediction.

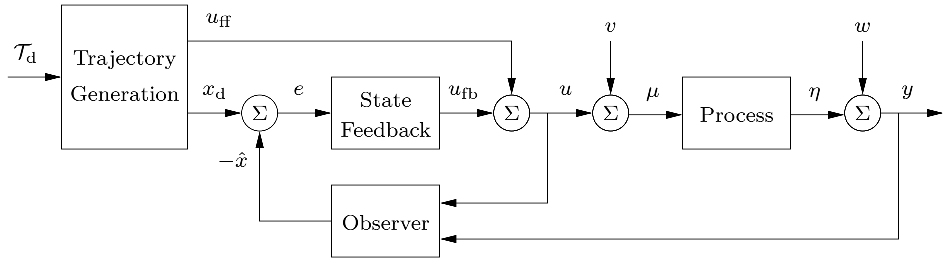


Figure 8.11: Block diagram of a controller based on a structure with two degrees of freedom that combines feedback and feedforward. The controller consists of a trajectory generator, state feedback, and an observer. The trajectory generation subsystem computes a feedforward command $u_{\rm ff}$ along with the desired state $x_{\rm d}$. The state feedback controller uses the estimated state and desired state to compute a corrective input $u_{\rm fb}$.

Notice that the models we have used do not have a direct term in the output. The separation theorem does not hold in this case because the nature of the disturbances is then influenced by the feedback.

8.5 State Space Controller Design

State estimators and state feedback are important components of a controller. In this section, we will add feedforward to arrive at a general controller structure that appears in many places in control theory and is the heart of most modern control systems. We will also briefly sketch how computers can be used to implement a controller based on output feedback.

Two Degree-of-Freedom Controller Architecture

In this chapter and the previous one we have emphasized feedback as a mechanism for minimizing tracking error; reference values were introduced simply by adding them to the state feedback through a gain $k_{\rm f}$. A more sophisticated way of doing this is shown by the block diagram in Figure 8.11, where the controller consists of three parts: an observer that computes estimates of the states based on a model and measured process inputs and outputs, a state feedback, and a trajectory generator that computes the desired behavior of all states $x_{\rm d}$ and a feedforward signal $u_{\rm ff}$. Under the ideal conditions of no disturbances and no modeling errors the signal $u_{\rm ff}$ generates the desired behavior $x_{\rm d}$ when applied to the process. The signals $x_{\rm d}$ and $u_{\rm ff}$ are generated from the task description $\mathcal{T}_{\rm d}$, which can represent different types of command signals depending on the application. In simple cases the task description is simply the reference signal r, and $x_{\rm d}$ and $u_{\rm ff}$ are generated by sending r through linear systems. For motion control problems, such as vehicle steering and robotics, the task description consists of the coordinates of a number of points (waypoints) that the vehicle should pass. In other situations the task description could be to transition from one state to another while optimizing some criterion.

To get some insight into the behavior of the system, consider the case when there are no disturbances and the system is in equilibrium with a constant reference signal and with the observer state $\hat{x}$ equal to the process state x. When the reference signal is changed, the signals u_{ff} and x_{d} will change. The observer tracks the state perfectly because the initial state was correct. The estimated state $\hat{x}$ is thus equal to the desired state x_{d}, and the feedback signal $u_{\mathrm{fb}} = K(x_{\mathrm{d}} - \hat{x})$ will also be zero. All action is thus created by the signals from the trajectory generator. If there are some disturbances or some modeling errors, the feedback signal will attempt to correct the situation.

This controller is said to have *two degrees of freedom* because the responses to reference signals and disturbances are decoupled. Disturbance responses are governed by the observer and the state feedback, while the response to command signals is governed by the trajectory generator (feedforward).

Feedforward Design and Trajectory Generation

We will now discuss design of controllers with the architecture shown in Figure 8.11. For an analytic description we start with the full nonlinear dynamics of the process

$$\frac{dx}{dt} = f(x,u), \qquad y = h(x,u). \tag{8.34}$$

A *feasible trajectory* for the system (8.34) is a pair $(x_{\mathrm{d}}(t), u_{\mathrm{ff}}(t))$ that satisfies the differential equation and generates the desired trajectory:

$$\frac{dx_{\mathrm{d}}(t)}{dt} = f\bigl(x_{\mathrm{d}}(t), u_{\mathrm{ff}}(t)\bigr), \qquad r(t) = h\bigl(x_{\mathrm{d}}(t), u_{\mathrm{ff}}(t)\bigr).$$

The problem of finding a feasible trajectory for a system is called the *trajectory generation* problem, with x_{d} representing the desired state for the (nominal) system and u_{ff} representing the desired input or the feedforward control. If we can find a feasible trajectory for the system, we can search for controllers of the form $u = \alpha(x, x_{\mathrm{d}}, u_{\mathrm{ff}})$ that track the desired reference trajectory.

In many applications, it is possible to attach a cost function to trajectories that describe how well they balance trajectory tracking with other factors, such as the magnitude of the inputs required. In such applications, it is natural to ask that we find the *optimal* controller with respect to some cost function:

$$\min_{u(\,\cdot\,)} \int_0^T L(x,u)\,dt + V\bigl(x(T)\bigr),$$

subject to the constraint

$$\dot{x} = f(x,u), \qquad x \in \mathbb{R}^n,\ u \in \mathbb{R}^p.$$

Abstractly, this is a constrained optimization problem where we seek a feasible trajectory $(x_{\mathrm{d}}(t), u_{\mathrm{ff}}(t))$ that minimizes the cost function. Depending on the form of the dynamics, this problem can be quite complex to solve, but there are good numerical packages for solving such problems, including handling constraints on the range of inputs as well as the allowable values of the state.

In some situations we can simplify the approach of generating feasible trajectories by exploiting the structure of the system. The next example illustrates one such approach.

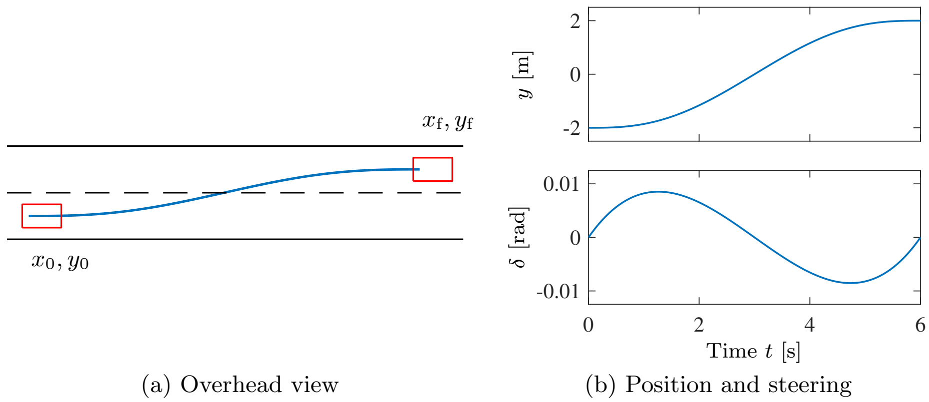


Figure 8.12: Trajectory generation for changing lanes. We wish to change from the right lane to the left lane over a distance of 90 m in 6 s. The planned trajectory in the xy plane is shown in (a) and the lateral position y and the steering angle δ over the maneuver time interval are shown in (b).

Example 8.8 Vehicle steering

To illustrate how we can use a two degree-of-freedom design to improve the performance of the system, consider the problem of steering a car to change lanes on a road, as illustrated in Figure 8.12a.

We use the non-normalized form of the dynamics, which were derived in Example 3.11. As shown in Exercise 3.6, using the center of the rear wheels as the reference ($\alpha = 0$) the dynamics can be written as

$$\frac{dx}{dt} = v\cos\theta, \qquad \frac{dy}{dt} = v\sin\theta, \qquad \frac{d\theta}{dt} = \frac{v}{b}\tan\delta,$$

where v is the forward velocity of the vehicle, θ is the heading angle, and δ is the steering angle. To generate a trajectory for the system, we note that we can solve for the states and inputs of the system given $x(t)$, $y(t)$ by solving the following sets of equations:

$$\begin{aligned} \dot{x} &= v\cos\theta, & \ddot{x} &= \dot{v}\cos\theta - v\dot{\theta}\sin\theta, \\ \dot{y} &= v\sin\theta, & \ddot{y} &= \dot{v}\sin\theta + v\dot{\theta}\cos\theta, \\ \dot{\theta} &= (v/b)\tan\delta. \end{aligned} \tag{8.35}$$

This set of five equations has five unknowns (θ, $\dot{\theta}$, v, $\dot{v}$ and δ) that can be solved using trigonometry and linear algebra given the path variables $x(t)$, $y(t)$ and their time derivatives. It follows that we can compute a feasible state trajectory for the system given any path $x(t)$, $y(t)$. (This special property of a system is known as *differential flatness* and is described in more detail below.)

To find a trajectory from an initial state (x_0, y_0, θ_0) to a final state (x_f, y_f, θ_f) at a time T, we look for a path $x(t), y(t)$ that satisfies

$$\begin{aligned} x(0) &= x_0, & x(T) &= x_\text{f}, \\ y(0) &= y_0, & y(T) &= y_\text{f}, \\ \dot{x}(0)\sin\theta_0 - \dot{y}(0)\cos\theta_0 &= 0, & \dot{x}(T)\sin\theta_\text{f} - \dot{y}(T)\cos\theta_\text{f} &= 0, \\ \dot{y}(0)\sin\theta_0 + \dot{x}(0)\cos\theta_0 &= v_0, & \dot{y}(T)\sin\theta_\text{f} + \dot{x}(T)\cos\theta_\text{f} &= v_\text{f}, \end{aligned} \tag{8.36}$$

where v_0 is the initial velocity and v_f is the final velocity along the trajectory. One such trajectory can be found by choosing $x(t)$ and $y(t)$ to have the form

$$x_\text{d}(t) = \alpha_0 + \alpha_1 t + \alpha_2 t^2 + \alpha_3 t^3, \qquad y_\text{d}(t) = \beta_0 + \beta_1 t + \beta_2 t^2 + \beta_3 t^3.$$

Substituting these equations into equation (8.36), we are left with a set of linear equations that can be solved for α_i, β_i, $i = 0, 1, 2, 3$. This gives a feasible trajectory for the system by using equation (8.35) to solve for θ_d, v_d, and δ_d.

Figure 8.12b shows a sample trajectory generated by a set of higher-order equations that also set the initial and final steering angle to zero. Notice that the feedforward input is different from zero, allowing the controller to command a steering angle that executes the turn in the absence of errors. ∇

The concept of differential flatness that we exploited in the previous example is a fairly general one and can be applied to many interesting trajectory generation problems. A nonlinear system (8.34) is *differentially flat* if there exists a flat output z such that the state x and the input u can be expressed as functions of the flat output z and a finite number of its derivatives:

$$x = \beta(z, \dot{z}, \dots, z^{(m)}), \qquad u = \gamma(z, \dot{z}, \dots, z^{(m)}). \tag{8.37}$$

The number of flat outputs is always equal to the number of system inputs. The vehicle steering model is differentially flat with the position of the rear wheels as the flat output.

A broad class of systems that is differentially flat is the class of reachable linear systems. For the linear system given in equation (7.6), which is in reachable canonical form, we have

$$\begin{gathered} z_1 = z_n^{(n-1)}, \quad z_2 = z_n^{(n-2)}, \quad \dots, \quad z_{n-1} = \dot{z}_n, \\ u = z_n^{(n)} + a_1 z_n^{(n-1)} + a_2 z_n^{(n-2)} + \dots + a_n z_n, \end{gathered}$$

and the nth component z_n of the state vector is thus a flat output. Since any reachable system can be transformed to reachable canonical form, it follows that every reachable linear system is differentially flat.

Note that no differential equations need to be integrated in order to compute the feasible trajectories for a differentially flat system (unlike optimal control methods, which often involve parameterizing the *input* and then solving the differential equations). The practical implication is that nominal trajectories and inputs that satisfy the equations of motion for a differentially flat system can be computed efficiently. The concept of differential flatness is described in more detail in the review article by Fliess et al. [FLMR95].

Disturbance Modeling and State Augmentation

We often have some information about load disturbances: they can be unknown constants, drifting with unknown rates, sinusoidal with known or unknown frequency, or stochastic signals. This information can be used by modeling the disturbances by differential equations and augmenting the process state with the disturbance states as was done in Section 7.4 and Example 8.6. We illustrate with a simple example.

Example 8.9 Integral action by state augmentation
Consider the system (8.1) and assume that there is a constant but unknown disturbance z acting additively on the process input. The system and the disturbance can then be modeled by augmenting the state x with z. An unknown constant can be modeled by the differential equation $dz/dt = 0$ and we obtain the following model for the process and its environment:

$$\frac{d}{dt}\begin{pmatrix} x \\ z \end{pmatrix} = \begin{pmatrix} A & B \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ z \end{pmatrix} + \begin{pmatrix} B \\ 0 \end{pmatrix} u, \qquad y = \begin{pmatrix} C & 0 \end{pmatrix}\begin{pmatrix} x \\ z \end{pmatrix}.$$

Notice that the disturbance state z is not reachable from u, but because the disturbance enters at the process input it can be attenuated by the control law

$$u = -K\hat{x} - \hat{z}, \tag{8.38}$$

where $\hat{x}$ and $\hat{z}$ are estimates of the state x and the disturbance z. The estimated disturbance can be obtained from the observer:

$$\frac{d\hat{x}}{dt} = A\hat{x} + B\hat{z} + Bu + L_x(y - C\hat{x}), \qquad \frac{d\hat{z}}{dt} = L_z(y - C\hat{x}).$$

Integrating the last equation and inserting the expression for $\hat{z}$ in the control law (8.38) gives

$$u = -K\hat{x} - L_z\int_0^t (y(\tau) - C\hat{x}(\tau))d\tau,$$

which is a state feedback controller with integral action. Notice that the integral action is created through estimation of a disturbance state. ∇

The idea of the example can be extended to many types of disturbances and we emphasized that much can be gained from modeling a process and its environment (disturbances acting on the process and measurement noise).

Feedback Design and Gain Scheduling

We now assume that the trajectory generator is able to compute a desired trajectory $(x_\mathrm{d}, u_\mathrm{ff})$ that satisfies the dynamics (8.34) and satisfies $r = h(x_\mathrm{d}, u_\mathrm{ff})$. To design the feedback controller, we construct the error system. Let $\xi = x - x_\mathrm{d}$ and $u_\mathrm{fb} = u - u_\mathrm{ff}$ and compute the dynamics for the error:

$$\begin{aligned}\dot{\xi} &= \dot{x} - \dot{x}_\mathrm{d} = f(x, u) - f(x_\mathrm{d}, u_\mathrm{ff}) \\ &= f(\xi + x_\mathrm{d}, v + u_\mathrm{ff}) - f(x_\mathrm{d}, u_\mathrm{ff}) =: F(\xi, v, x_\mathrm{d}(t), u_\mathrm{ff}(t)).\end{aligned}$$

For trajectory tracking, we can assume that ξ is small (if our controller is doing a good job), and so we can linearize around $\xi = 0$:

$$\frac{d\xi}{dt} \approx A(t)\xi + B(t)v, \quad h(x,u) \approx C(t)x(t)$$
$$A(t) = \left.\frac{\partial F}{\partial \xi}\right|_{(x_\mathrm{d}(t),u_\mathrm{ff}(t))}, \quad B(t) = \left.\frac{\partial F}{\partial v}\right|_{(x_\mathrm{d}(t),u_\mathrm{ff}(t))}, \quad C(t) = \left.\frac{\partial h}{\partial \xi}\right|_{(x_\mathrm{d}(t),u_\mathrm{ff}(t))}.$$

In general, this system is time-varying. Note that ξ corresponds to $-e$ in Figure 8.11 due to the convention of using negative feedback in the block diagram. We can now proceed to use LQR to compute the time-varying feedback gain $K(t) = Q_u^{-1}(t)B^T(t)S(t)$ by solving the Riccati differential equation (7.31) and the Kalman filter gain $L(t) = P(t)C^T(t)R_w^{-1}(t)$, where $P(t)$ is obtained by solving the Riccati equation (8.30).

Assume now that x_d and u_ff are either constant or slowly varying (with respect to the process dynamics). It is often the case that $A(t)$, $B(t)$, and $C(t)$ depend only on x_d, in which case it is convenient to write $A(t) = A(x_\mathrm{d})$, $B(t) = B(x_\mathrm{d})$, and $C(t) = C(x_\mathrm{d})$. This allows us to consider just the linear system given by $A(x_\mathrm{d})$, $B(x_\mathrm{d})$, and $C(x_\mathrm{d})$. If we design a state feedback controller $K(x_\mathrm{d})$ for each x_d, then we can regulate the system using the feedback

$$u_\mathrm{fb} = -K(x_\mathrm{d})\xi.$$

Substituting back the definitions of ξ and u_fb, our controller becomes

$$u = u_\mathrm{fb} + u_\mathrm{ff} = -K(x_\mathrm{d})(x - x_\mathrm{d}) + u_\mathrm{ff}.$$

This form of controller is called a *gain scheduled* linear controller with *feedforward* u_ff.

Example 8.10 Steering control with velocity scheduling

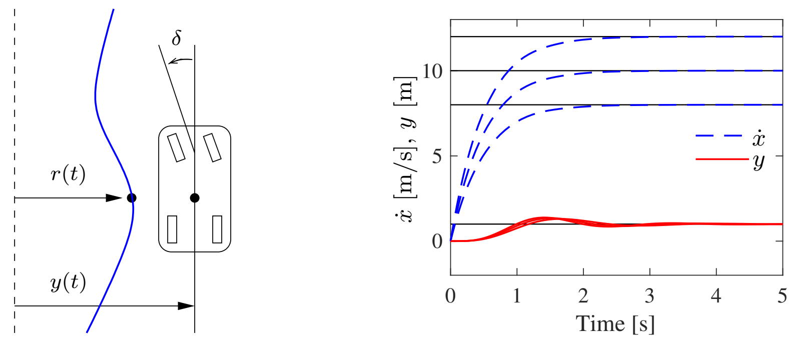
Consider the problem of controlling the motion of a automobile so that it follows a given trajectory on the ground, as shown in Figure 8.13a. We use the model derived in Example 8.8. A simple feasible trajectory for the system is to follow a straight line in the x direction at lateral position y_r and fixed velocity v_r. This corresponds to a desired state $x_\mathrm{d} = (v_\mathrm{r}t, y_\mathrm{r}, 0)$ and nominal input $u_\mathrm{ff} = (v_\mathrm{r}, 0)$. Note that $(x_\mathrm{d}, u_\mathrm{ff})$ is not an equilibrium point for the full system, but it does satisfy the equations of motion.

Linearizing the system about the desired trajectory, we obtain

$$A_\mathrm{d} = \left.\frac{\partial f}{\partial x}\right|_{(x_\mathrm{d},u_\mathrm{ff})} = \left.\begin{pmatrix} 0 & 0 & -\sin\theta \\ 0 & 0 & \cos\theta \\ 0 & 0 & 0 \end{pmatrix}\right|_{(x_\mathrm{d},u_\mathrm{ff})} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix},$$
$$B_\mathrm{d} = \left.\frac{\partial f}{\partial u}\right|_{(x_\mathrm{d},u_\mathrm{ff})} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & v_\mathrm{r}/l \end{pmatrix}.$$

We form the error dynamics by setting $e = x - x_\mathrm{d}$ and $w = u - u_\mathrm{ff}$:

$$\frac{de_x}{dt} = w_1, \qquad \frac{de_y}{dt} = e_\theta, \qquad \frac{de_\theta}{dt} = \frac{v_\mathrm{r}}{l}w_2.$$



(a) Vehicle configuration (b) Controller response

Figure 8.13: Vehicle steering using gain scheduling. (a) Vehicle configuration consists of the x, y position of the vehicle, its angle with respect to the road, and the steering wheel angle. (b) Step responses for the vehicle lateral position (solid) and forward velocity (dashed). Gain scheduling is used to set the feedback controller gains for the different forward velocities.

We see that the first state is decoupled from the second two states and hence we can design a controller by treating these two subsystems separately. Suppose that we wish to place the closed loop eigenvalues of the longitudinal dynamics (e_x) at λ_1 and place the closed loop eigenvalues of the lateral dynamics (e_y, e_θ) at the roots of the polynomial equation $s^2 + a_1 s + a_2 = 0$. This can be accomplished by setting

$$w_1 = -\lambda_1 e_x, \qquad w_2 = \frac{l}{v_\mathrm{r}}(a_1 e_y + a_2 e_\theta).$$

Note that the gain l/v_r depends on the velocity v_r (or equivalently on the nominal input u_ff), giving us a gain scheduled controller.

In the original inputs and state coordinates, the controller has the form

$$\begin{pmatrix} v \\ \delta \end{pmatrix} = -\underbrace{\begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \dfrac{a_1 l}{v_\mathrm{r}} & \dfrac{a_2 l}{v_\mathrm{r}} \end{pmatrix}}_{K_\mathrm{d}} \underbrace{\begin{pmatrix} x - v_\mathrm{r} t \\ y - y_\mathrm{r} \\ \theta \end{pmatrix}}_{e} + \underbrace{\begin{pmatrix} v_\mathrm{r} \\ 0 \end{pmatrix}}_{u_\mathrm{ff}}.$$

The form of the controller shows that at low speeds the gains in the steering angle will be high, meaning that we must turn the wheel harder to achieve the same effect. As the speed increases, the gains become smaller. This matches the usual experience that at high speed a very small amount of actuation is required to control the lateral position of a car. Note that the gains go to infinity when the vehicle is stopped ($v_\mathrm{r} = 0$), corresponding to the fact that the system is not reachable at this point.

Figure 8.13b shows the response of the controller to a step change in lateral position at three different reference speeds. Notice that the rate of the response is constant, independent of the reference speed, reflecting the fact that the gain scheduled controllers each set the closed loop eigenvalues to the same values. ∇

Nonlinear Estimation

Finally, we briefly comment on the observer represented in Figure 8.11 for the case where the process dynamics are not necessarily linear. Since we are now considering a nonlinear system that operates over a wide range of the state space, it is desirable to use full nonlinear dynamics for the prediction portion of the observer. This can then be combined with a linear correction term, so that the observer has the form:

$$\frac{d\hat{x}}{dt} = f(\hat{x}, u) + L(\hat{x})(y - h(\hat{x})).$$

The estimator gain $L(\hat{x})$ is the observer gain obtained by linearizing the system around the currently estimated state. This form of the observer is known as an *extended Kalman filter* and has proved to be a very effective means of estimating the state of a nonlinear system.

The combination of trajectory generation, trajectory tracking, and nonlinear estimation provides a means for state space control of nonlinear systems. There are many ways to generate the feedforward signal, and there are also many different ways to compute the feedback gain K and the observer gain L. Note that once again the internal model principle applies: the overall controller contains a model of the system to be controlled and its environment through the observer.

Computer Implementation

The controllers obtained so far have been described by ordinary differential equations. They can be implemented directly using analog components, whether electronic circuits, hydraulic valves, or other physical devices. Since in modern engineering applications most controllers are implemented using computers, we will briefly discuss how this can be done.

A computer-controlled system typically operates periodically: every cycle, signals from the sensors are sampled and converted to digital form by an analog-to-digital (A/D) converter, the control signal is computed and the resulting output is converted to analog form for the actuators, as shown in Figure 8.14. To illustrate the main principles of how to implement feedback in this environment, we consider the controller described by equations (8.15) and (8.16), i.e.,

$$\frac{d\hat{x}}{dt} = A\hat{x} + Bu + L(y - C\hat{x}), \qquad u = -K\hat{x} + k_\mathrm{f} r.$$

The second equation consists only of additions and multiplications and can thus be implemented directly on a computer. The first equation can be implemented by approximating the derivative by a difference

$$\frac{d\hat{x}}{dt} \approx \frac{\hat{x}(t_{k+1}) - \hat{x}(t_k)}{h} = A\hat{x}(t_k) + Bu(t_k) + L\big(y(t_k) - C\hat{x}(t_k)\big),$$

where t_k are the sampling instants and $h = t_{k+1} - t_k$ is the sampling period. Rewriting the equation to isolate $\hat{x}(t_{k+1})$, we get the difference equation

$$\hat{x}(t_{k+1}) = \hat{x}(t_k) + h\Big(A\hat{x}(t_k) + Bu(t_k) + L\big(y(t_k) - C\hat{x}(t_k)\big)\Big). \tag{8.39}$$

The calculation of the estimated state at time t_{k+1} requires only addition and multiplication and can easily be done by a computer. A section of pseudocode for the program that performs this calculation is as follows.

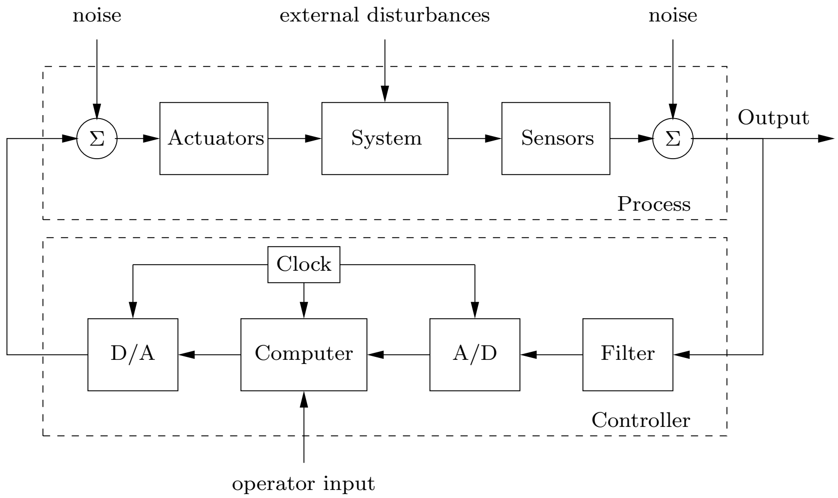


Figure 8.14: Components of a computer-controlled system. The controller consists of analog-to-digital (A/D) and digital-to-analog (D/A) converters, as well as a computer that implements the control algorithm. A system clock controls the operation of the controller, synchronizing the A/D, D/A, and computing processes. The operator input is also fed to the computer as an external input.

```
% Control algorithm - main loop
r = adin(ch1)                       % read reference
y = adin(ch2)                       % get process output
(xd, uff) = trajgen(r, t)           % generate feedforward
u = K*(xd - xhat) + uff             % compute control variable
daout(ch1, u)                       % set analog output
xhat = xhat + h*(A*x+B*u+L*(y-C*x)) % update state estimate
```

The program runs periodically at a fixed sampling period h. Notice that the number of computations between reading the analog input and setting the analog output has been minimized by updating the state after the analog output has been set. The program has an array of states `xhat` that represents the state estimate. The choice of sampling period requires some care.

There are more sophisticated ways of approximating a differential equation by a difference equation. If the control signal is constant between the sampling instants, it is possible to obtain exact equations; see [ÅW97].

There are several practical issues that also must be dealt with. For example, it is necessary to filter measured signals before they are sampled so that the filtered signal has little frequency content above $f_s/2$ (the Nyquist frequency), where $f_s = 1/h$ is the sampling frequency. This avoids a phenomenon known as *aliasing*. If controllers with integral action are used, it is also necessary to provide protection so that the integral does not become too large when the actuator saturates. This issue, called *integrator windup*, is studied in more detail in Chapter 11. Care must also be taken so that parameter changes do not cause disturbances.

8.6 Further Reading

The notion of observability is due to Kalman [Kal61b] and, combined with the dual notion of reachability, it was a major stepping stone toward establishing state space control theory beginning in the 1960s. The observer first appeared as the Kalman filter in the paper by Kalman [Kal61a] for the discrete-time case and Kalman and Bucy [KB61] for the continuous-time case. Kalman also conjectured that the controller for output feedback could be obtained by combining a state feedback with a Kalman filter; see the quote in the beginning of this chapter. This result, which is known as the *separation theorem*, is mathematically subtle. Attempts of proof were made by Josep and Tou [JT61] and Gunckel and Franklin [GF71], but a rigorous proof was given by Georgiou and Lindquist [GL13] in 2013. The combined result is known as the linear quadratic Gaussian control theory; a compact treatment is given in the books by Anderson and Moore [AM90], Åström [Åst06], and Lindquist and Picci [LP15]. It was also shown that solutions to robust control problems had a similar structure but with different ways of computing observer and state feedback gains [DGKF89]. The importance of systems with two degrees of freedom that combine feedback and feedforward was emphasized by Horowitz [Hor63]. The controller structure discussed in Section 8.5 is based on these ideas. The particular form in Figure 8.11 appeared in [ÅW97], where computer implementation of the controller was discussed in detail. The hypothesis that motion control in humans is based on a combination of feedback and feedforward was proposed by Ito in 1970 [Ito70]. Differentially flat systems were originally studied by Fliess et al. [FLMR92]; they are very useful for trajectory generation.

Exercises

8.1 (Coordinate transformations) Consider a system under a coordinate transformation $z = Tx$, where $T \in \mathbb{R}^{n\times n}$ is an invertible matrix. Show that the observability matrix for the transformed system is given by $\widetilde{W}_o = W_o T^{-1}$ and hence observability is independent of the choice of coordinates.

8.2 Show that the system depicted in Figure 8.2 is not observable.

8.3 (Multi-input, multi-output observability) Consider the multi-input, multi-output system given by

$$\frac{dx}{dt} = Ax + Bu, \qquad y = Cx,$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^p$, and $y \in \mathbb{R}^q$. Show that the states can be determined from the input u and the output y and their derivatives if the observability matrix W_o given by equation (8.4) has n independent rows.

8.4 (Observable canonical form) Show that if a system is observable, then there exists a change of coordinates $z = Tx$ that puts the transformed system into observable canonical form.

8.5 (Bicycle dynamics) The linearized model for a bicycle is given in equation (4.5), which has the form

$$J\frac{d^2\varphi}{dt^2} - \frac{Dv_0}{b}\frac{d\delta}{dt} = mgh\varphi + \frac{mv_0^2 h}{b}\delta,$$

where φ is the tilt of the bicycle and δ is the steering angle. Give conditions under which the system is observable and explain any special situations where it loses observability.

8.6 (Observer design by eigenvalue assignment) Consider the system

$$\frac{dx}{dt} = Ax = \begin{pmatrix} -1 & 0 \\ 1 & 0 \end{pmatrix} x + \begin{pmatrix} a-1 \\ 1 \end{pmatrix} u, \qquad y = Cx = \begin{pmatrix} 0 & 1 \end{pmatrix} x.$$

Design an observer such that $\det(sI - LC) = s^2 + 2\zeta_o\omega_o s + \omega_o^2$ with values $\omega_o = 10$ and $\zeta_o = 0.6$.

8.7 (Vectored thrust aircraft) The lateral dynamics of the vectored thrust aircraft example described in Example 7.9 can be obtained by considering the motion described by the states $z = (x, \theta, \dot{x}, \dot{\theta})$. Construct an estimator for these dynamics by setting the eigenvalues of the observer into a *Butterworth pattern* with $\lambda_{\text{bw}} = -3.83 \pm 9.24i$, $-9.24 \pm 3.83i$. Using this estimator combined with the state space controller computed in Example 7.9, plot the step response of the closed loop system.

8.8 (Observer for Teorell's compartment model) Teorell's compartment model, shown in Figure 4.17, has the following state space representation:

$$\frac{dx}{dt} = \begin{pmatrix} -k_1 & 0 & 0 & 0 & 0 \\ k_1 & -k_2 - k_4 & 0 & k_3 & 0 \\ 0 & k_4 & 0 & 0 & 0 \\ 0 & k_2 & 0 & -k_3 - k_5 & 0 \\ 0 & 0 & 0 & k_5 & 0 \end{pmatrix} x + \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} u,$$

where representative parameters are $k_1 = 0.02$, $k_2 = 0.1$, $k_3 = 0.05$, $k_4 = k_5 = 0.005$. The concentration of a drug that is active in compartment 5 is measured in the bloodstream (compartment 2). Determine the compartments that are observable from measurement of concentration in the bloodstream and design an estimator for these concentrations base on eigenvalue assignment. Choose the closed loop eigenvalues -0.03, -0.05, and -0.1. Simulate the system when the input is a pulse injection.

8.9 (Whipple bicycle model) Consider the Whipple bicycle model given by equation (4.8) in Section 4.2. A state feedback for the system was designed in Exercise 7.12. Design an observer and an output feedback for the system.

8.10 (Kalman decomposition) Consider a linear system characterized by the matrices

$$A = \begin{pmatrix} -2 & 1 & -1 & 2 \\ 1 & -3 & 0 & 2 \\ 1 & 1 & -4 & 2 \\ 0 & 1 & -1 & -1 \end{pmatrix}, \quad B = \begin{pmatrix} 2 \\ 2 \\ 2 \\ 1 \end{pmatrix}, \quad C = \begin{pmatrix} 0 & 1 & -1 & 0 \end{pmatrix}, \quad D = 0.$$

Construct a Kalman decomposition for the system. (Hint: Try to diagonalize.)

8.11 (Kalman filtering for a first-order system) Consider the system

$$\frac{dx}{dt} = ax + v, \qquad y = cx + w$$

where all variables are scalar. The signals v and w are uncorrelated white noise disturbances with zero mean values and covariance functions

$$\mathbb{E}(v(s)v^T(t)) = r_v\delta(t-s), \quad \mathbb{E}(w(s)w^T(t)) = r_w\delta(t-s).$$

The initial condition is Gaussian with mean value x_0 and covariance P_0. Determine the Kalman filter for the system and analyze what happens for large t.

8.12 (Vertical alignment) In navigation systems it is important to align a system to the vertical. This can be accomplished by measuring the vertical acceleration and controlling the platform so that the measured acceleration is zero. A simplified one-dimensional version of the problem can be modeled by

$$\frac{d\varphi}{dt} = u, \qquad u = -ky, \qquad y = \varphi + w,$$

where φ is the alignment error, u the control signal, y the measured signal, and w the measurement noise, which is assumed to be white noise with zero mean and covariance function $\mathbb{E}(w(s)w^T(t)) = r_w\delta(t-s)$. The initial misalignment is assumed to be a random variable with zero mean and the covariance P_0. Determine a time-varying gain $k(t)$ such that the error goes to zero as fast as possible. Compare this with a constant gain.