Chapter 7

Output Feedback

There are none.

Abstract for "Gauranteed Margins for LQG Regulators", John Doyle, 1978 [Doy78].

In the last chapter we considered the use of state feedback to modify the dynamics of a system through feedback. In many applications, it is not practical to measure all of the states directly and we can measure only a small number of outputs (corresponding to the sensors that are available). In this chapter we show how to use output feedback to modify the dynamics of the system, through the use of observers (also called "state estimators"). We introduce the concept of observability and show that if a system is observable, it is possible to recover the state from measurements of the inputs and outputs to the system.

7.1 Observability

In Section 6.2 of the previous chapter it was shown that it is possible to find a feedback that gives desired closed loop eigenvalues provided that the system is reachable and that all states are measured. For many situations, it is highly unrealistic to assume that all states are measured. In this section we will investigate how the state can be estimated by using a mathematical model and a few measurements. It will be shown that the computation of the states can be done by a dynamical system called an *observer*.

Consider a system described by

$$\begin{aligned} \frac{dx}{dt} &= Ax + Bu \\ y &= Cx + Du, \end{aligned} \tag{7.1}$$

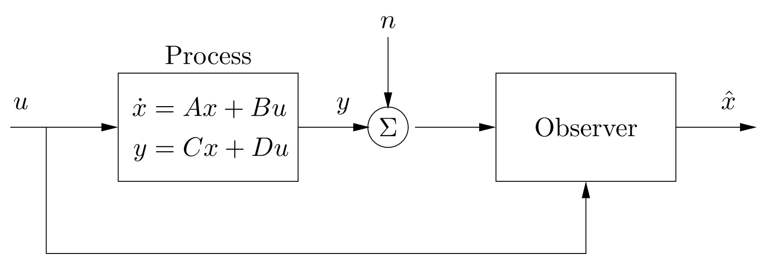

Figure 7.1: Block diagram for an observer.

where $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}$ the input, and $y \in \mathbb{R}$ the measured output. We wish to estimate the state of the system from its inputs and outputs, as illustrated in Figure 7.1. We assume that there is only one measured signal, i.e. that the signal y is a scalar and that C is a (row) vector. This signal may be corrupted by noise, n, although we shall start by considering the noise-free case. We write $\hat{x}$ for the state estimate given by the observer.

Definition 7.1 (Observability)**.** A linear system is *observable* if for any $T > 0$ it is possible to determine the state of the system $x(T)$ through measurements of $y(t)$ and $u(t)$ on the interval $[0, T]$.

The problem of observability is one that has many important applications, even outside of feedback systems. If a system is observable, then there are no "hidden" dynamics inside it; we can understand everything that is going on through observation (over time) of the inputs and outputs. As we shall see, the problem of observability is of significant practical interest because it will tell if a set of sensors are sufficient for controlling a system. Sensors combined with a mathematical model can also be viewed as a "virtual sensor" that gives information about variables that are not measured directly. The definition above holds for nonlinear systems as well, and the results discussed here have extensions to the nonlinear case.

When discussing reachability in the last chapter we neglected the output and focused on the state. Similarly, it is convenient here to initially neglect the input and focus on the system

$$\begin{aligned} \frac{dx}{dt} &= Ax \\ y &= Cx. \end{aligned} \tag{7.2}$$

We wish to understand when it is possible to determine the state from observations of the output.

The output itself gives the projection of the state on vectors that are rows of the matrix C. The observability problem can immediately be solved if the matrix C is invertible. If the matrix is not invertible we can take derivatives of the output to obtain

$$\frac{dy}{dt} = C\frac{dx}{dt} = CAx.$$

From the derivative of the output we thus get the projection of the state on vectors which are rows of the matrix CA. Proceeding in this way we get

$$\begin{pmatrix} y \\ \dot{y} \\ \ddot{y} \\ \vdots \\ y^{(n-1)} \end{pmatrix} = \begin{pmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{n-1} \end{pmatrix} x. \tag{7.3}$$

We thus find that the state can be determined if the matrix

$$W_o = \begin{pmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{n-1} \end{pmatrix} \tag{7.4}$$

has n independent rows. It turns out that we need not consider any derivatives higher than $n-1$ (this is an application of the Cayley-Hamilton theorem [Str88]).

The calculation can easily be extended to systems with inputs. The state is then given by a linear combination of inputs and outputs and their higher derivatives. We leave this as an exercise for the reader.

In practice, differentiation can give very large errors when there is measurement noise and therefore the method sketched above is not particularly practical. We will address this issue in more detail in the next section, but for now we have the following basic result:

Theorem 7.1. *A linear system of the form* (7.1) *is observable if and only if the observability matrix W_o is full rank.*

Proof. The sufficiency of the observability rank condition follows from the analysis above. To prove necessity, suppose that the system is observable but W_o is not full rank. Let $v \in \mathbb{R}^n$, $v \neq 0$ be a vector in the null space of W_o, so that $W_o v = 0$. If we let $x(0) = v$ be the initial condition for the system and choose $u = 0$, then the output is given by $y(t) = Ce^{At}v$. Since e^{At} can be written as a power series in A and since A^n and higher powers can be rewritten in terms of lower powers of A (by the Cayley-Hamilton theorem), it follows that the output will be identically zero (the reader should fill in the missing steps if this is not clear). However, if both the input and output of the system are 0, then a valid estimate of the state is $\hat{x} = 0$ for all time, which is clearly incorrect since $x(0) = v \neq 0$. Hence by contradiction we must have that W_o is full rank if the system is observable. □

Example 7.1 (Bicycle dynamics)**.** To demonstrate the concept of observability, we consider the bicycle system, introduced in Section 3.2. Consider the linearized model for the dynamics in equation (3.5), which has the form

$$J\frac{d^2\varphi}{dt^2} - \frac{Dv_0}{b}\frac{d\delta}{dt} = mgh\varphi + \frac{mv_0^2 h}{b}\delta,$$

where φ is the tilt of the bicycle and δ is the steering angle. Taking the torque on the handle bars as an input and the lateral deviation as the output, we can write the dynamics in state space form as (Exercise 3.3)

$$\begin{aligned}
\frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} &= \begin{pmatrix} 0 & mgh/J \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} u \\
y &= \begin{pmatrix} \frac{Dv_0}{bJ} & \frac{mv_0^2 h}{bJ} \end{pmatrix} x.
\end{aligned}$$

The observability of this system determines whether it is possible to determine the entire system state (tilt angle and tilt rate) from observations of the input (steering angle) and output (vehicle position).

The observability matrix is

$$W_0 = \begin{pmatrix} \dfrac{Dv_0}{bJ} & \dfrac{mv_0^2 h}{bJ} \\ \dfrac{mv_0^2 h}{bJ} & \dfrac{mgh}{J} \cdot \frac{Dv_0}{bJ} \end{pmatrix}$$

and its determinant is

$$\det W_o = \left(\frac{Dv_0}{bJ}\right)^2 \frac{mgh}{J} - \left(\frac{mv_0^2 h}{bJ}\right)^2.$$

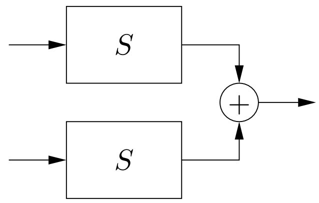

Figure 7.2: A non-observable system.

Under most choices of parameters, the determinant will be nonzero and hence the system is observable. However, if the parameters of the system are chosen such that

$$\frac{mv_0h}{D} = \sqrt{\frac{mgh}{J}}$$

then we see that W_o becomes singular and the system is not observable. This case is explored in more detail in the exercises. ∇

Example 7.2 (Unobservable systems)**.** It is useful to have an understanding of the mechanisms that make a system unobservable. Such a system is shown in Figure 7.2. The system is composed of two identical systems whose outputs are added. It seems intuitively clear that it is not possible to deduce the states from the output since we cannot deduce the individual output contributions from the sum. This can also be seen formally (Exercise 1). ∇

As in the case of reachability, certain canonical forms will be useful in studying observability. We define the observable canonical form to be the dual of the reachable canonical form.

Definition 7.2 (Observable canonical form)**.** A linear state space system is in *observable canonical form* if its dynamics are given by

$$\frac{dz}{dt} = \begin{pmatrix} -a_1 & 1 & 0 & \cdots & 0 \\ -a_2 & 0 & 1 & & 0 \\ \vdots & & & \ddots & \\ -a_{n-1} & 0 & 0 & & 1 \\ -a_n & 0 & 0 & \cdots & 0 \end{pmatrix} z + \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_{n-1} \\ b_n \end{pmatrix} u$$

$$y = \begin{pmatrix} 1 & 0 & 0 \cdots & 0 \end{pmatrix} z + Du.$$

Figure 7.3 shows a block diagram for a system in observable canonical form. As in the case of reachable canonical form, we see that the coeffi-

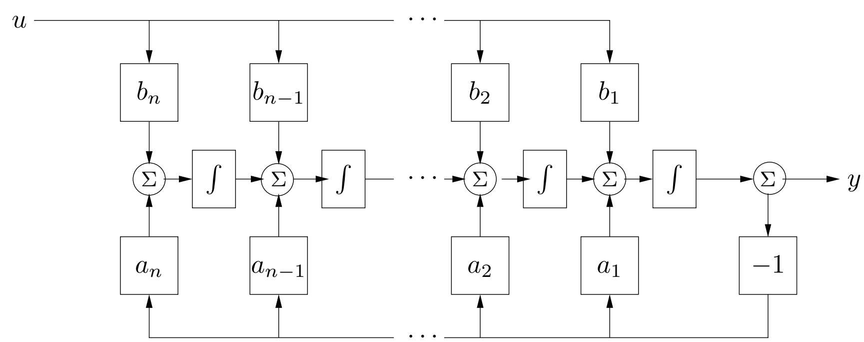

Figure 7.3: Block diagram of a system on observable canonical form.

cients in the system description appear directly in the block diagram. The characteristic equation for a system in observable canonical form is given by

$$\lambda(s) = s^n + a_1 s^{n-1} + \cdots + a_{n-1}s + a_n. \tag{7.5}$$

It is possible to reason about the observability of a system in observable canonical form by studying the block diagram. If the input u and the output are available the state x_1 can clearly be computed. Differentiating x_1 we also obtain the input to the integrator that generates x_1 and we can now obtain $x_2 = \dot{x}_1 + a_1 x_1 - b_1 u$. Proceeding in this way we can clearly compute all states. The computation will however require that the signals are differentiated.

We can now proceed with a formal analysis. The observability matrix for a system in observable canonical form is given by

$$W_o = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ -a_1 & 1 & 0 & \dots & 0 \\ -a_1^2 - a_1 a_2 & -a_1 & 1 & \dots & 0 \\ \vdots & & & \ddots & \vdots \\ * & & & \dots & 1 \end{pmatrix},$$

where * represents as entry whose exact value is not important. The rows of this matrix are linearly independent (since it is lower triangular) and hence W_o is full rank. A straightforward but tedious calculation shows that the inverse of the observability matrix has a simple form, given by

$$W_o^{-1} = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ a_1 & 1 & 0 & \cdots & 0 \\ a_2 & a_1 & 1 & \cdots & 0 \\ \vdots & & & \ddots & \\ a_{n-1} & a_{n-2} & a_{n-3} & \cdots & 1 \end{pmatrix}.$$

As in the case of reachability, it turns out that if a system is observable then there always exists a transformation T that converts the system into reachable canonical form (Exercise 3). This is very useful for proofs, since it lets us assume that a system is in reachable canonical form without any loss of generality.

7.2 State Estimation

Having defined the concept of observability, we now return to the question of how to construct an observer for a system. We will look for observers that can be represented as a linear dynamical system that takes the inputs and outputs of the system we are observing and produces an estimate of the system's state. That is, we wish to construct a dynamical system of the form

$$\frac{d\hat{x}}{dt} = F\hat{x} + Gu + Hy,$$

where u and y are the input and output of the original system and $\hat{x} \in \mathbb{R}^n$ is an estimate of the state with the property that $\hat{x}(t) \to x(t)$ as $t \to \infty$.

The Basic Observer

For a system governed by equation (7.1), we can attempt to determine the state simply by simulating the equations with the correct input. An estimate of the state is then given by

$$\frac{d\hat{x}}{dt} = A\hat{x} + Bu. \tag{7.6}$$

To find the properties of this estimate, introduce the estimation error

$$\tilde{x} = x - \hat{x}.$$

It follows from equations (7.1) and (7.6) that

$$\frac{d\tilde{x}}{dt} = A\tilde{x}.$$

If matrix A has all its eigenvalues in the left half plane, the error $\tilde{x}$ will thus go to zero and hence equation (7.6) is a dynamical system whose output converges to the state of the system (7.1).

The observer given by equation (7.6) uses only the process input u; the measured signal does not appear in the equation. We must also require that the system is stable and essentially our estimator converges because the state of both the observer and the estimator are going zero. This is not very useful in a control design context since we want to have our estimate converge quickly to a nonzero state, so that we can make use of it in our controller. We will therefore attempt to modify the observer so that the output is used and its convergence properties can be designed to be fast relative to the system's dynamics. This version will also work for unstable systems.

Consider the observer

$$\frac{d\hat{x}}{dt} = A\hat{x} + Bu + L(y - C\hat{x}). \tag{7.7}$$

This can be considered as a generalization of equation (7.6). Feedback from the measured output is provided by adding the term $L(y - C\hat{x})$, which is proportional to the difference between the observed output and the output that is predicted by the observer. To investigate the observer (7.7), form the error $\tilde{x} = x - \hat{x}$. It follows from equations (7.1) and (7.7) that

$$\frac{d\tilde{x}}{dt} = (A - LC)\tilde{x}.$$

If the matrix L can be chosen in such a way that the matrix $A - LC$ has eigenvalues with negative real parts, the error $\tilde{x}$ will go to zero. The convergence rate is determined by an appropriate selection of the eigenvalues.

The problem of determining the matrix L such that $A - LC$ has prescribed eigenvalues is very similar to the eigenvalue assignment problem that was solved in the previous chapter. In fact, since the eigenvalues of the matrix and its transpose are the same, it is equivalent to search for L^T such that $A^T - C^T L^T$ has the desired eigenvalues. This is precisely the eigenvalue assignment problem that we solved in the previous chapter, with $\tilde{A} = A^T$, $\tilde{B} = C^T$ and $\tilde{K} = L^T$. Thus, using the results of Theorem 6.3, we can have the following theorem on observer design:

Theorem 7.2 (Observer design by eigenvalue assignment)**.** *Consider the system given by*

$$\begin{aligned} \frac{dx}{dt} &= Ax + Bu \\ y &= Cx \end{aligned} \tag{7.8}$$

with one input and one output. Let $\lambda(s) = s^n + a_1 s^{n-1} + \cdots + a_{n-1}s + a_n$ *be the characteristic polynomial for* A*. If the system is observable then the dynamical system*

$$\frac{d\hat{x}}{dt} = A\hat{x} + Bu + L(y - C\hat{x}) \tag{7.9}$$

is an observer for the system, with L *chosen as*

$$L = W_o^{-1}\tilde{W}_o \begin{pmatrix} p_1 - a_1 \\ p_2 - a_2 \\ \vdots \\ p_n - a_n \end{pmatrix}, \tag{7.10}$$

and the matrices W_o *and* $\tilde{W}_o$ *given by*

$$W_o = \begin{pmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{pmatrix} \qquad \tilde{W}_o = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ a_1 & 1 & 0 & \cdots & 0 \\ a_2 & a_1 & 1 & \cdots & 0 \\ \vdots & & & & \\ a_{n-1} & a_{n-2} & a_{n-3} & \cdots & 1 \end{pmatrix}^{-1}.$$

The resulting observer error $\tilde{x} = x - \hat{x}$ *is governed by a differential equation having the characteristic polynomial*

$$p(s) = s^n + p_1 s^{n-1} + \cdots + p_n.$$

The dynamical system (7.9) is called an observer for (the states of) the system (7.8) because it will generate an approximation of the states of the system from its inputs and outputs. This particular form of an observer is a much more useful form than the one given by pure differentiation in equation (7.3).

Interpretation of the Observer

The observer is a dynamical system whose inputs are the process input u and process output y. The rate of change of the estimate is composed of two terms. One term, $A\hat{x} + Bu$, is the rate of change computed from the model with $\hat{x}$ substituted for x. The other term, $L(y - \hat{y})$, is proportional to the difference $e = y - \hat{y}$ between measured output y and its estimate $\hat{y} = C\hat{x}$. The estimator gain L is a matrix that tells how the error e is weighted and distributed among the states. The observer thus combines measurements

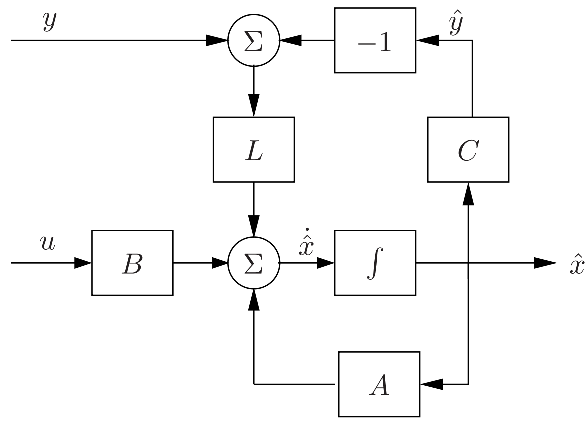

Figure 7.4: Block diagram of the observer. Notice that the observer contains a copy of the process.

with a dynamical model of the system. A block diagram of the observer is shown in Figure 7.4.

Notice the similarity between the problems of finding a state feedback and finding the observer. The key is that both of these problems are equivalent to the same algebraic problem. In eigenvalue assignment it is attempted to find K so that $A - BK$ has given eigenvalues. For the observer design it is instead attempted to find L so that $A - LC$ has given eigenvalues. The following equivalence can be established between the problems:

$$\begin{aligned} A &\leftrightarrow A^T & \qquad K &\leftrightarrow L^T \\ B &\leftrightarrow C^T & \qquad W_r &\leftrightarrow W_o^T \end{aligned}$$

The observer design problem is often called the *dual* of the state feedback design problem. The similarity between design of state feedback and observers also means that the same computer code can be used for both problems.

Computing the Observer Gain

The observer gain can be computed in several different ways. For simple problems it is convenient to introduce the elements of L as unknown parameters, determine the characteristic polynomial of the observer and identify it with the desired characteristic polynomial. Another alternative is to use the fact that the observer gain can be obtained by inspection if the system is in observable canonical form. The observer gain is then obtained by transformation to the canonical form. There are also reliable numerical algorithms, which are identical to the algorithms for computing the state feedback. The procedures are illustrated by example.

Example 7.3 (Vehicle steering)**.** Consider the normalized, linear model for vehicle steering in Example 5.12. The dynamics relating steering angle u to lateral path deviation y is given by the state space model

$$\begin{aligned}\frac{dx}{dt} &= \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} x + \begin{pmatrix} \alpha \\ 1 \end{pmatrix} u \\ y &= \begin{pmatrix} 1 & 0 \end{pmatrix} x.\end{aligned} \tag{7.11}$$

Recall that the state x_1 represents the lateral path deviation and that x_2 represents turning rate. We will now derive an observer that uses the system model to determine turning rate from the measured path deviation.

The observability matrix is

$$W_o = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

i.e., the identity matrix. The system is thus observable and the eigenvalue assignment problem can be solved. We have

$$A - LC = \begin{pmatrix} -l_1 & 1 \\ -l_2 & 0 \end{pmatrix},$$

which has the characteristic polynomial

$$\det(sI - A + LC) = \det \begin{pmatrix} s + l_1 & -1 \\ l_2 & s \end{pmatrix} = s^2 + l_1 s + l_2.$$

Assuming that it is desired to have an observer with the characteristic polynomial

$$s^2 + p_1 s + p_2 = s^2 + 2\zeta_o \omega_o s + \omega_o^2,$$

the observer gains should be chosen as

$$\begin{aligned} l_1 &= p_1 = 2\zeta_o\omega_o \\ l_2 &= p_2 = \omega_o^2. \end{aligned}$$

The observer is then

$$\frac{d\hat{x}}{dt} = A\hat{x} + Bu + L(y = Cx) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \hat{x} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u + \begin{pmatrix} l_1 \\ l_2 \end{pmatrix} (y - \hat{x}_1).$$

∇

For larger systems, the `place` or `acker` commands can be used in MATLAB. Note that these functions are the same as the ones used for eigenvalue assignment with state feedback; for estimator design, one simply uses the transpose of the dynamics matrix and the output matrix.

7.3 Control using Estimated State

In this section we will consider the same system as in the previous sections, i.e., the state space system described by

$$\begin{aligned} \frac{dx}{dt} &= Ax + Bu \\ y &= Cx. \end{aligned} \tag{7.12}$$

We wish to design a feedback controller for the system where only the output is measured. As before, we will be assume that u and y are scalars. We also assume that the system is reachable and observable. In Chapter 6 we found a feedback of the form

$$u = Kx + k_r r$$

for the case that all states could be measured and in Section 7.2 we have developed an observer that can generate estimates of the state $\hat{x}$ based on inputs and outputs. In this section we will combine the ideas of these sections to find a feedback that gives desired closed loop eigenvalues for systems where only outputs are available for feedback.

If all states are not measurable, it seems reasonable to try the feedback

$$u = -K\hat{x} + k_r r \tag{7.13}$$

where $\hat{x}$ is the output of an observer of the state, i.e.

$$\frac{d\hat{x}}{dt} = A\hat{x} + Bu + L(y - C\hat{x}). \tag{7.14}$$

Since the system (7.12) and the observer (7.14) both are of state dimension n, the closed loop system has state dimension $2n$. The states of the combined system are x and $\hat{x}$. The evolution of the states is described by equations (7.12), (7.13) and (7.14). To analyze the closed loop system, the state variable $\hat{x}$ is replaced by

$$\tilde{x} = x - \hat{x}. \tag{7.15}$$

Subtraction of equation (7.14) from equation (7.12) gives

$$\frac{d\tilde{x}}{dt} = Ax - A\hat{x} - L(y - C\hat{x}) = A\tilde{x} - LC\tilde{x} = (A - LC)\tilde{x}.$$

Returning to the process dynamics, introducing u from equation (7.13) into equation (7.12) and using equation (7.15) to eliminate $\hat{x}$ gives

$$\begin{aligned}\frac{dx}{dt} &= Ax + Bu = Ax - BK\hat{x} + Bk_r r = Ax - BK(x - \tilde{x}) + Bk_r r\\ &= (A - BK)x + BK\tilde{x} + Bk_r r.\end{aligned}$$

The closed loop system is thus governed by

$$\frac{d}{dt}\begin{pmatrix} x \\ \tilde{x} \end{pmatrix} = \begin{pmatrix} A - BK & BK \\ 0 & A - LC \end{pmatrix}\begin{pmatrix} x \\ \tilde{x} \end{pmatrix} + \begin{pmatrix} Bk_r \\ 0 \end{pmatrix} r. \tag{7.16}$$

Notice that the state $\tilde{x}$, representing the observer error, is not affected by the command signal r. This is desirable since we do not want the reference signal to generate observer errors.

Since the dynamics matrix is block diagonal, we find that the characteristic polynomial of the closed loop system is

$$\lambda(s) = \det(sI - A + BK)\det(sI - A + LC).$$

This polynomial is a product of two terms: the characteristic polynomial of the closed loop system obtained with state feedback and the characteristic polynomial of the observer error. The feedback (7.13) that was motivated heuristically thus provides a very neat solution to the eigenvalue assignment problem. The result is summarized as follows.

Theorem 7.3 (Eigenvalue assignment by output feedback)**.** *Consider the system*

$$\begin{aligned}\frac{dx}{dt} &= Ax + Bu\\ y &= Cx.\end{aligned}$$

The controller described by

$$\begin{aligned}u &= -K\hat{x} + k_r r\\ \frac{d\hat{x}}{dt} &= A\hat{x} + Bu + L(y - C\hat{x})\end{aligned}$$

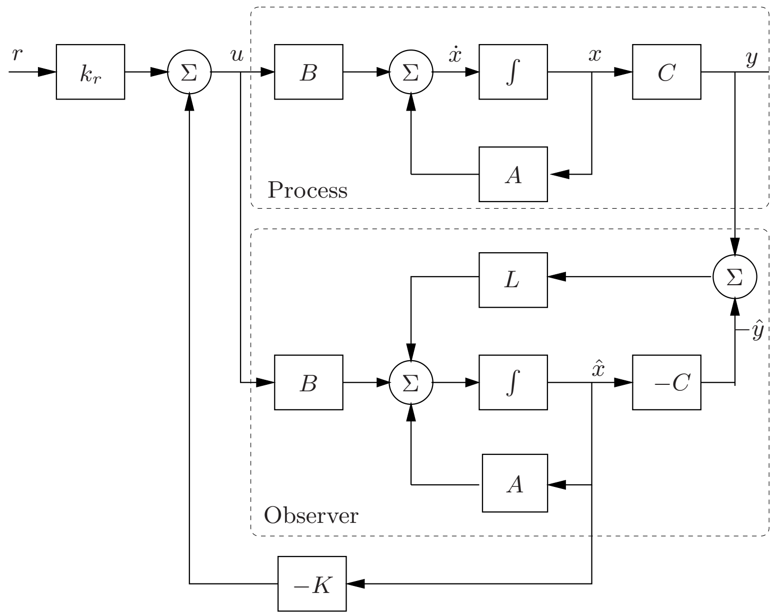

Figure 7.5: Block diagram of a control system that combines state feedback with an observer.

gives a closed loop system with the characteristic polynomial

$$\lambda(s) = \det(sI - A + BK)\det(sI - A + LC).$$

This polynomial can be assigned arbitrary roots if the system is reachable and observable.

The controller has a strong intuitive appeal: it can be thought of as composed of two parts, one state feedback and one observer. The feedback gain K can be computed as if all state variables can be measured. This property is called the *separation principle* and it allows us to independently solve for the state space controller and the state space estimator.

A block diagram of the controller is shown in Figure 7.5. Notice that the controller contains a dynamical model of the plant. This is called the *internal model principle*: the controller contains a model of the process being controlled. Indeed, the dynamics of the controller is due to the observer and can thus be viewed as a dynamical system with input y and output u:

$$\begin{aligned} \frac{d\hat{x}}{dt} &= (A - BK - LC)\hat{x} + Ly \\ u &= -K\hat{x} + k_r r. \end{aligned} \tag{7.17}$$

Example 7.4 (Vehicle steering)**.** Consider again the normalized, linear model for vehicle steering in Example 5.12. The dynamics relating steering angle u to lateral path deviation y is given by the state space model (7.11). Combining the state feedback derived in Example 6.4 with the observer determined in Example 7.3 we find that the controller is given by

$$\begin{aligned} \frac{d\hat{x}}{dt} &= A\hat{x} + Bu + L(y - Cx) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \hat{x} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u + \begin{pmatrix} l_1 \\ l_2 \end{pmatrix} (y - \hat{x}_1) \\ u &= -K\hat{x} + k_r r = k_1(r - x_1) - k_2 x_2 \end{aligned}$$

The controller is thus a dynamical system of second order. Elimination of the variable u gives

$$\begin{aligned} \frac{d\hat{x}}{dt} &= (A - BK - LC)\hat{x} + Ly + Bk_r r \\ &= \begin{pmatrix} -l_1 - \alpha k_1 & 1 - \alpha k_2 \\ -k_1 - l_2 & -k_2 \end{pmatrix} \hat{x} + \begin{pmatrix} l_1 \\ l_2 \end{pmatrix} y + \begin{pmatrix} \alpha \\ 1 \end{pmatrix} k_1 r \\ u &= -K\hat{x} + k_r r = -\begin{pmatrix} k_1 & k_2 \end{pmatrix} \hat{x} + k_1 r. \end{aligned}$$

The controller is a dynamical system of second order, with two inputs y and r and one output u. ∇

7.4 Kalman Filtering

One of the principal uses of observers in practice is to estimate the state of a system in the presence of *noisy* measurements. We have not yet treated noise in our analysis and a full treatment of stochastic dynamical systems is beyond the scope of this text. In this section, we present a brief introduction to the use of stochastic systems analysis for constructing observers. We work primarily in discrete time to avoid some of the complications associated with continuous time random processes and to keep the mathematical prerequisites to a minimum. This section assumes basic knowledge of random variables and stochastic processes.

Consider a discrete time, linear system with dynamics

$$\begin{aligned} x_{k+1} &= Ax_k + Bu_k + Fv_k \\ y_k &= Cx_k + w_k, \end{aligned} \tag{7.18}$$

where v_k and w_k are Gaussian, white noise processes satisfying

$$
\begin{aligned}
&E\{v_k\} = 0 && E\{w_k\} = 0 \\
&E\{v_k v_j^T\} = \begin{cases} 0 & k \neq j \\ R_v & k = j \end{cases} && E\{w_k w_j^T\} = \begin{cases} 0 & k \neq j \\ R_w & k = j \end{cases} \\
&&E\{v_k w_j^T\} = 0.
\end{aligned} \tag{7.19}
$$

We assume that the initial condition is also modeled as a Gaussian random variable with

$$E\{x_0\} = x_0 \qquad E\{x_0 x_0^T\} = P_0. \tag{7.20}$$

We wish to find an estimate $\hat{x}_k$ that minimizes the mean square error $E\{(x_k - \hat{x}_k)(x_k - \hat{x}_k)^T\}$ given the measurements $\{y(\delta) : 0 \leq \tau \leq t\}$. We consider an observer in the same basic form as derived previously:

$$\hat{x}_{k+1} = A\hat{x}_k + Bu_k + L_k(y_k - C\hat{x}_k). \tag{7.21}$$

The following theorem summarizes the main result.

Theorem 7.4. *Consider a random process x_k with dynamics* (7.18) *and noise processes and initial conditions described by equations* (7.19) *and* (7.20). *The observer gain L that minimizes the mean square error is given by*

$$L_k = A^T P_k C^T (R_w + C P_k C^T)^{-1},$$

where

$$
\begin{aligned}
P_{k+1} &= (A - LC)P_k(A - LC)^T + R_v + LR_w L^T \\
P_0 &= E\{X(0)X^T(0)\}.
\end{aligned} \tag{7.22}
$$

Before we prove this result, we reflect on its form and function. First, note that the Kalman filter has the form of a *recursive* filter: given $P_k = E\{E_k E_k^T\}$ at time k, can compute how the estimate and covariance *change*. Thus we do not need to keep track of old values of the output. Furthermore, the Kalman filter gives the estimate $\hat{x}_k$ *and* the covariance $P_{E,k}$, so we can see how reliable the estimate is. It can also be shown that the Kalman filter extracts the maximum possible information about output data. If we form the residual between the measured output and the estimated output,

$$e_k = y_k - C\hat{x}_k,$$

we can can show that for the Kalman filter the correlation matrix is

$$R_e(j,k) = W\delta_{jk}.$$

In other words, the error is a white noise process, so there is no remaining dynamic information content in the error.

In the special case when the noise is stationary (Q, R constant) and *if* P_k converges, then the observer gain is constant:

$$K = A^T PC^T(R_w + CPC^T),$$

where

$$P = APA^T + R_v - APC^T\big(R_w + CPC^T\big)^{-1}CPA^T.$$

We see that the optimal gain depends on both the process noise and the measurement noise, but in a nontrivial way. Like the use of LQR to choose state feedback gains, the Kalman filter permits a systematic derivation of the observer gains given a description of the noise processes. The solution for the constant gain case is solved by the `dlqe` command in MATLAB.

Proof (of theorem). We wish to minimize the mean square of the error, $E\{(x_k - \hat{x}_k)(x_k - \hat{x}_k)^T\}$. We will define this quantity as P_k and then show that it satisfies the recursion given in equation (7.22). By definition,

$$\begin{aligned} P_{k+1} &= E\{x_{k+1}x_{k+1}^T\} \\ &= (A - LC)P_k(A - LC)^T + R_v + LR_wL^T \\ &= AP_kA^T - AP_kC^TL^T - LCA^T + L(R_w + CP_kC^T)L^T \end{aligned}$$

Letting $R_\epsilon = (R_w + CP_kC^T)$, we have

$$\begin{aligned} P_{k+1} &= AP_kA^T - AP_kC^TL^T - LCA^T + LR_\epsilon L^T \\ &= AP_kA^T + \big(L - AP_kC^TR_\epsilon^{-1}\big)R_\epsilon\big(L - AP_KC^TR_\epsilon^{-1}\big)^T \\ &\quad - AP_kC^TR_\epsilon^{-1}CP_k^TA^T + R_w. \end{aligned}$$

In order to minimize this expression, we choose $L = AP_kC^TR_\epsilon^{-1}$ and the theorem is proven. □

The Kalman filter can also be applied to continuous time stochastic processes. The mathematical derivation of this result requires more sophisticated tools, but the final form of the estimator is relatively straightforward.

Consider a continuous stochastic system

$$\begin{aligned} \dot{x} &= Ax + Bu + Fv \qquad & E\{v(s)v^T(t)\} &= Q(t)\delta(t-s) \\ y &= Cx + w \qquad & E\{w(s)w^T(t)\} &= R(t)\delta(t-s) \end{aligned}$$

Assume that the disturbance v and noise w are zero-mean and Gaussian (but not necessarily stationary):

$$\begin{aligned}\text{pdf}(v) &= \frac{1}{\sqrt[n]{2\pi}\sqrt{\det Q}} e^{-\frac{1}{2}v^T Q^{-1} v} \\ \text{pdf}(w) &= \dots \quad \text{(using R)}\end{aligned}$$

We wish to find the estimate $\hat{x}(t)$ that minimizes the mean square error $E\{(x(t) - \hat{x}(t))(x(t) - \hat{x}(t))^T\}$ given $\{y(\tau) : 0 \le \tau \le t\}$.

Theorem 7.5 (Kalman-Bucy, 1961)**.** *The optimal estimator has the form of a linear observer*

$$\dot{\hat{x}} = A\hat{x} + Bu + L(y - C\hat{x})$$

where $L(t) = P(t)C^T R^{-1}$ *and* $P(t) = E\{(x(t) - \hat{x}(t))(x(t) - \hat{x}(t))^T\}$ *and satisfies*

$$\begin{aligned}\dot{P} &= AP + PA^T - PC^T R^{-1}(t) CP + FQ(t)F^T \\ P(0) &= E\{x(0)x^T(0)\}\end{aligned}$$

7.5 State Space Control Systems

In this section we consider a collection of additional topics on the design and analysis of control systems using state space tools.

Computer Implementation

The controllers obtained so far have been described by ordinary differential equations. They can be implemented directly using analog components, whether electronic circuits, hydraulic valves or other physical devices. Since in modern engineering applications most controllers are implemented using computers we will briefly discuss how this can be done.

A computer controlled system typically operates periodically: every cycle, signals from the sensors are sampled and converted to digital form by the A/D converter, the control signal is computed, and the resulting output is converted to analog form for the actuators (as shown in Figure 1.3 on page 5). To illustrate the main principles of how to implement feedback in this environment, we consider the controller described by equations (7.13) and (7.14), i.e.,

$$\begin{aligned}u &= -K\hat{x} + k_r r \\ \frac{d\hat{x}}{dt} &= A\hat{x} + Bu + K(y - C\hat{x}).\end{aligned}$$

The first equation consists only of additions and multiplications and can thus be implemented directly on a computer. The second equation has to be approximated. A simple way is to approximate the derivative by a difference

$$\frac{dx}{dt} \approx \frac{\hat{x}(t_{k+1}) - \hat{x}(t_k)}{h} = A\hat{x}(t_k) + Bu(t_k) + K(y(t_k) - C\hat{x}(t_k))$$

where t_k are the sampling instants and $h = t_{k+1} - t_k$ is the sampling period. Rewriting the equation to isolate $x(t_{k+1})$, we get

$$\hat{x}(t_{k+1}) = \hat{x}(t_k) + h\big(A\hat{x}(t_k) + Bu(t_k) + K(y(t_k) - C\hat{x}(t_k))\big). \tag{7.23}$$

The calculation of the estimated state at time t_{k+1} only requires addition and multiplication and can easily be done by a computer. A section of pseudo code for the program that performs this calculation is

```
% Control algorithm - main loop
r = adin(ch1)                        % read setpoint from ch1
y = adin(ch2)                        % get process output from ch2
u = C*xhat + Kr*r                    % compute control variable
daout(ch1, u)                        % set analog output on ch1
xhat = xhat + h*(A*x+B*u+L*(y-C*x))  % update state estimate
```

The program runs periodically at a fixed rate h. Notice that the number of computations between reading the analog input and setting the analog output has been minimized. The state is updated after the analog output has been set. The program has an array of states, `xhat`, that represents the state estimate. The choice of sampling period requires some care.

There are several practical issues that also must be dealt with. For example it is necessary to filter a signal before it is sampled so that the filtered signal has little frequency content above $f_s/2$ where f_s is the sampling frequency. If controllers with integral action are used, it is also necessary to provide protection so that the integral does not become too large when the actuator saturates. This issue, called *integrator windup*, is studied in more detail in Chapter 10. Care must also be taken so that parameter changes do not cause disturbances.

A General Controller Structure

We now consider a general control structure that pulls together the various results the the previous and current chapters. This structure is one that appears in may places in control theory and is the heart of most modern control systems.

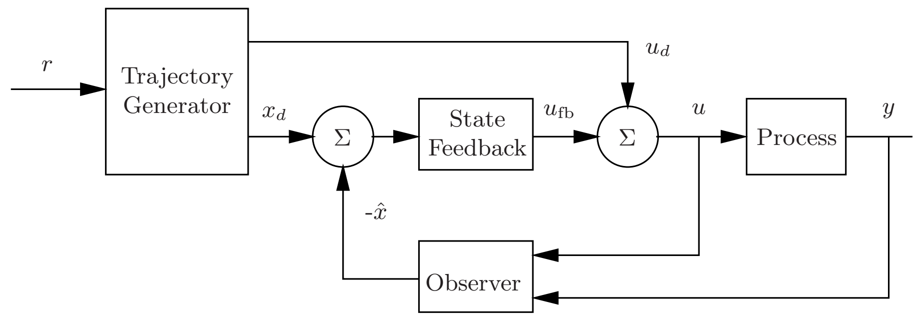

Figure 7.6: Block diagram of a controller based on a structure with two degrees of freedom. The controller consists of a command signal generator, state feedback and an observer.

We begin by generalizing the way we handle the reference input. So far reference signals have been introduced simply by adding them to the state feedback through a gain k_r. A more sophisticated way of doing this is shown by the block diagram in Figure 7.6, where the controller consists of three parts: an observer that computes estimates of the states based on a model and measured process inputs and outputs, a state feedback and a trajectory generator that generates the desired behavior of all states x_d and a feedforward signal u_d. The signal u_d is such that it generates the desired behavior of the states when applied to the system, under the ideal conditions of no disturbances and no modeling errors. The controller is said to have *two degrees of freedom* because the response to command signals and disturbances are decoupled. Disturbance responses are governed by the observer and the state feedback and the response to command signals is governed by the trajectory generator (feedfoward).

We start with the full nonlinear dynamics of the process

$$\begin{aligned} \dot{x} &= f(x,u) \\ y &= h(x,u). \end{aligned} \tag{7.24}$$

Assume that the trajectory generator is able to generate a desired trajectory (x_d, u_d) that satisfies the dynamics (7.24) and satisfies $r = h(x_d, u_d)$. To design the controller, we construct the error system. We will assume for simplicity that $f(x,u) = f(x) + g(x)u$ (i.e., the system is nonlinear in the state, but linear in the input; this is often the case in applications). Let

$e = x - x_d$, $v = u - u_d$ and compute the dynamics for the error:

$$\begin{aligned}\dot{e} = \dot{x} - \dot{x}_d &= f(x) + g(x)u - f(x_d) + g(x_d)u_d \\ &= f(e + x_d) - f(x_d) + g(e + x_d)(v + u_d) - g(x_d)u_d \\ &= F(e, v, x_d(t), u_d(t))\end{aligned}$$

In general, this system is time varying.

For trajectory tracking, we can assume that e is small (if our controller is doing a good job) and so we can linearize around $e = 0$:

$$\dot{e} \approx A(t)e + B(t)v$$

where

$$A(t) = \left.\frac{\partial F}{\partial e}\right|_{(x_d(t),u_d(t))} \qquad B(t) = \left.\frac{\partial F}{\partial v}\right|_{(x_d(t),u_d(t)}.$$

It is often the case that $A(t)$ and $B(t)$ depend only on x_d, in which case it is convenient to write $A(t) = A(x_d)$ and $B(t) = B(x_d)$.

Assume now that x_d and u_d are either constant or slowly varying (with respect to the performance criterion). This allows us to consider just the (constant) linear system given by $(A(x_d), B(x_d))$. If we design a state feedback controller $K(x_d)$ for each x_d, then we can regulate the system using the feedback

$$v = K(x_d)e.$$

Substituting back the definitions of e and v, our controller becomes

$$u = K(x_d)(x - x_d) + u_d$$

This form of controller is called a *gain scheduled* linear controller with *feedforward* u_d.

Finally, we consider the observer. We can use the full nonlinear dynamics for the prediction portion of the observer and the linearized system for the correction term:

$$\dot{\hat{x}} = f(\hat{x}, u) + L(\hat{x})(y - h(\hat{x}, u))$$

where $L(\hat{x})$ is the observer gain obtained by linearizing the system around the currently estimate state. This form of the observer is known as an *extended Kalman filter* and has proven to be a very effective means of estimating the state of a nonlinear system.

To get some insight into the overall behavior of the system, we consider what happens when the command signal is changed. To fix the ideas let us assume that the system is in equilibrium with the observer state $\hat{x}$ equal to the process state x. When the command signal is changed a feedforward signal $u_d(t)$ is generated. This signal has the property that the process output gives the desired state $x_d(t)$ when the feedforward signal is applied to the system. The process state changes in response to the feedforward signal. The observer tracks the state perfectly because the initial state was correct. The estimated state $\hat{x}$ will thus be equal to the desired state x_d and the feedback signal $L(x_d - \hat{x})$ is zero. If there are some disturbances or some modeling errors the feedback signal will be different from zero and attempt to correct the situation.

The controller given in Figure 7.6 is a very general structure. There are many ways to generate the feedforward signal and there are also many different ways to compute the feedback gain K and the observer gain L. Note that once again the internal model principle applies: the controller contains a model of the system to be controlled.

The Kalman Decomposition

In this chapter and the previous one, we have seen that two fundamental properties of a linear input/output system are reachability and observability. It turns out that these two properties can be used to classify the dynamics of a system. The key result is Kalman's decomposition theorem, which says that a linear system can be divided into four subsystems: $\mathcal{S}_{ro}$ which is reachable and observable, $\mathcal{S}_{r\bar{o}}$ which is reachable but not observable, $\mathcal{S}_{\bar{r}o}$ which is not reachable but is observable, and $\mathcal{S}_{\bar{r}\bar{o}}$ which is neither reachable nor observable.

We will first consider this in the special case of systems where the matrix A has distinct eigenvalues. In this case we can find a set of coordinates such that the A matrix is diagonal and, with some additional reordering of the states, the system can be written as

$$\frac{dz}{dt} = \begin{pmatrix} \Lambda_{ro} & 0 & 0 & 0 \\ 0 & \Lambda_{r\bar{o}} & 0 & 0 \\ 0 & 0 & \Lambda_{\bar{r}o} & 0 \\ 0 & 0 & 0 & \Lambda_{\bar{r}\bar{o}} \end{pmatrix} z + \begin{pmatrix} \beta_{ro} \\ \beta_{ro} \\ 0 \\ 0 \end{pmatrix} u$$

$$y = \begin{pmatrix} \gamma_{ro} & 0 & \gamma_{\bar{r}o} & 0 \end{pmatrix} z + Du.$$

All states z_k such that $\beta_k \neq 0$ are controllable and all states such that $\gamma_k \neq 0$ are observable. The frequency response of the system is given by

$$G(s) = \gamma_{ro}(sI - A_{ro})^{-1}\beta_{ro} + D$$

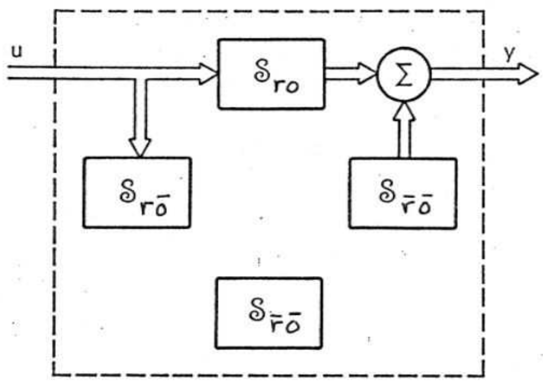

Figure 7.7: Kalman's decomposition of a linear system with distinct eigenvalues.

and it is uniquely given by the subsystem that is reachable and observable. Thus from the input/output point of view, it is only the reachable and observable dynamics that matter. A block diagram of the system illustrating this property is given in Figure 7.7.

The general case of the Kalman decomposition is more complicated and requires some additional linear algebra. Introduce the reachable subspace $\mathcal{X}_r$ which is the linear subspace spanned by the columns of the reachability matrix W_r. The state space is the direct sum of $\mathcal{X}_r$ and another linear subspace $\mathcal{X}_{\bar{r}}$. Notice that $\mathcal{X}_r$ is unique but that $\mathcal{X}_{\bar{r}}$ can be chosen in many different ways. Choosing coordinates with $x_r \in \mathcal{X}_r$ and $x_{\bar{r}} \in \mathcal{X}_{\bar{r}}$ the system equations can be written as

$$\frac{d}{dt}\begin{pmatrix} x_r \\ x_{\bar{r}} \end{pmatrix} = \begin{pmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{pmatrix}\begin{pmatrix} x_r \\ x_{\bar{r}} \end{pmatrix} + \begin{pmatrix} B_1 \\ 0 \end{pmatrix} u, \tag{7.25}$$

where the states x_r are reachable and $x_{\bar{r}}$ are non-reachable.

Introduce the unique subspace $\mathcal{X}_{\bar{o}}$ of non-observable states. This is the right null space of the observability matrix W_o. The state space is the direct sum of $\mathcal{X}_{\bar{o}}$ and the non-unique subspace $\mathcal{X}_o$. Choosing a coordinate system with $x_o \in \mathcal{X}_o$ and $x_{\bar{o}} \in \mathcal{X}_{\bar{o}}$ the system equations can be written as

$$\begin{aligned} \frac{d}{dt}\begin{pmatrix} x_o \\ x_{\bar{o}} \end{pmatrix} &= \begin{pmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{pmatrix}\begin{pmatrix} x_o \\ x_{\bar{o}} \end{pmatrix} \\ y &= \begin{pmatrix} C_1 & 0 \end{pmatrix}\begin{pmatrix} x_o \\ x_{\bar{o}} \end{pmatrix}, \end{aligned} \tag{7.26}$$

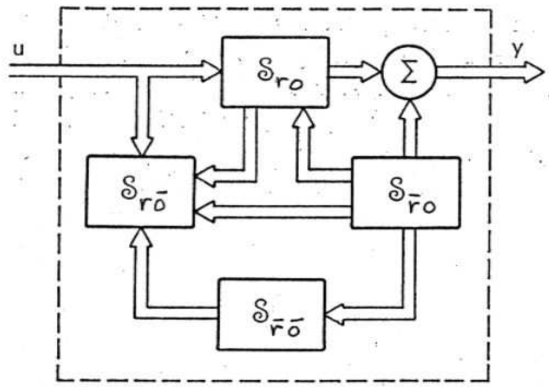

Figure 7.8: Kalman's decomposition of a linear system with general eigenvalues.

where the states x_o are observable and $x_{\bar{o}}$ are not observable.

The intersection of two linear subspaces is also a linear subspace. Introduce $\mathcal{X}_{r\bar{o}}$ as the intersection of $\mathcal{X}_r$ and $\mathcal{X}_{\bar{o}}$ and the complementary linear subspace $\mathcal{X}_{ro}$, which together with $\mathcal{X}_{r\bar{o}}$ spans $\mathcal{X}_r$. Finally, we introduce the linear subspace $\mathcal{X}_{\bar{r}o}$ which together with $\mathcal{X}_{r\bar{o}}$, $\mathcal{X}_{r\bar{o}}$ and $\mathcal{X}_{r\bar{o}}$ spans the full state space. Notice that the decomposition is not unique because only the subspace $\mathcal{X}_{r\bar{o}}$ is unique.

Combining the representations (7.25) and (7.26) we find that a linear system can be transformed to the form

$$\frac{dx}{dt} = \begin{pmatrix} A_{11} & 0 & A_{13} & 0 \\ A_{21} & A_{22} & A_{23} & A_{24} \\ 0 & 0 & A_{33} & 0 \\ 0 & 0 & A_{43} & A_{44} \end{pmatrix} x + \begin{pmatrix} B_1 \\ B_2 \\ 0 \\ 0 \end{pmatrix} u \tag{7.27}$$
$$y = \begin{pmatrix} C_1 & 0 & C_2 & 0 \end{pmatrix} x,$$

where the state vector has been partitioned as

$$x = \begin{pmatrix} x_{ro} \\ x_{r\bar{o}} \\ x_{\bar{r}o} \\ x_{\bar{r}\bar{o}} \end{pmatrix}$$

A block diagram of the system is shown in Figure 7.8. By tracing the arrows in the diagram we find that the input influences the systems $\mathcal{S}_{ro}$ and

$\mathcal{S}_{\bar{r}o}$ and that the output is influenced by $\mathcal{S}_{ro}$ and $\mathcal{S}_{\bar{r}o}$. The system $\mathcal{S}_{\bar{r}\bar{o}}$ is neither connected to the input nor the output. The frequency response of the system is thus

$$G(s) = C_1(sI - A_{11})^{-1}B_1, \tag{7.28}$$

which is the dynamics of the reachable and observable subsystem $\mathcal{S}_{ro}$.

7.6 Further Reading

The notion of observability is due to Kalman [Kal61b] and, combined with the dual notion of reachability, it was a major stepping stone toward establishing state space control theory beginning in the 1960s. For linear systems the output is a projection of the state and it may seem unnecessary to estimate the full state since a projection is already available. Luenberger [Lue71] constructed an reduced order observer that only reconstructs the state that is not measured directly.

The main result of this chapter is the general controller structure in Figure 7.6. This controller structure emerged as a result of solving optimization problems. The observer first appeared as the Kalman filter which was also the solution to an optimization problem [Kal61a, KB61]. It was then shown that the solution to an optimization with output feedback could be obtained by combining a state feedback with a Kalman filter [JT61, GF71]. Later it was found that the controller with the same structure also emerged as solutions of other simpler deterministic control problems like the ones discussed in this chapter [**?**, **?**]. Much later it was shown that solutions to robust control problems also had a similar structure but with different ways of computing observer and state feedback gains [DGKF89]. The material is now an essential part of the tools in control.

A more detailed presentation of stochastic control theory is given in [Åst70].

7.7 Exercises

1. Show that the system depicted in Figure 7.2 is not observable.

2. Consider a system under a coordinate transformation $z = Tx$, where $T \in \mathbb{R}^{n\times n}$ is an invertible matrix. Show that the observability matrix for the transformed system is given by $\tilde{W}_o = W_o T^{-1}$ and hence observability is independent of the choice of coordinates.

3. Show that if a system is observable, then there exists a change of coordinates $z = Tx$ that puts the transformed system into reachable canonical form.