Differential Forms and Stokes’ Theorem

Jerrold E. Marsden

Control and Dynamical Systems, Caltech
http://www.cds.caltech.edu/~marsden/
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Basic example: differential of a real-valued function.
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Basic example: differential of a real-valued function.

2-form \( \Omega \): a map \( \Omega(m) : T_mM \times T_mM \rightarrow \mathbb{R} \) that assigns to each point \( m \in M \) a skew-symmetric bilinear form on the tangent space \( T_mM \) to \( M \) at \( m \).
A \textbf{k-form} $\alpha$ (or \textit{differential form of degree k}) is a map

$$\alpha(m) : T_mM \times \cdots \times T_mM(k \text{ factors}) \to \mathbb{R},$$

which, for each $m \in M$, is a skew-symmetric $k$-multi-linear map on the tangent space $T_mM$ to $M$ at $m$. 
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It is \textit{skew} (or \textit{alternating}) when it changes sign whenever two of its arguments are interchanged.
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- Orientation or “handedness”
Let $x^1, \ldots, x^n$ denote coordinates on $M$, let

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be the corresponding basis for $T_m M$. 
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At each \( m \in M \), we can write a 2-form as
\[
\Omega_m(v, w) = \Omega_{ij}(m)v^i w^j,
\]
where
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\Omega_{ij}(m) = \Omega_m \left( \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j} \right),
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Similarly for $k$-forms.
If $\alpha$ is a $(0, k)$-tensor on a manifold $M$ and $\beta$ is a $(0, l)$-tensor, their tensor product (sometimes called the outer product), $\alpha \otimes \beta$ is the $(0, k + l)$-tensor on $M$ defined by

\[
(\alpha \otimes \beta)_m(v_1, \ldots, v_{k+l}) = \alpha_m(v_1, \ldots, v_k)\beta_m(v_{k+1}, \ldots, v_{k+l})
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at each point $m \in M$.

Outer product of two vectors is a matrix.
If $t$ is a $(0, p)$-tensor, define the \textit{alternation operator} $A$ acting on $t$ by

$$A(t)(v_1, \ldots, v_p) = \frac{1}{p!} \sum_{\pi \in S_p} \text{sgn}(\pi) t(v_{\pi(1)}, \ldots, v_{\pi(p)})$$

where \text{sgn}(\pi) is the \textit{sign} of the permutation $\pi$,

$$\text{sgn}(\pi) = \begin{cases} +1 & \text{if } \pi \text{ is even} , \\ -1 & \text{if } \pi \text{ is odd} \end{cases}$$

and $S_p$ is the group of all permutations of the set \{1, 2, \ldots, p\}. 
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and \( S_p \) is the group of all permutations of the set \( \{1, 2, \ldots, p\} \).

The operator \( A \) therefore \textit{skew-symmetrizes} \( p \)-multilinear maps.
If \( \alpha \) is a \( k \)-form and \( \beta \) is an \( l \)-form on \( M \), their *wedge product* \( \alpha \wedge \beta \) is the \( (k + l) \)-form on \( M \) defined by

\[
\alpha \wedge \beta = \frac{(k + l)!}{k! l!} A(\alpha \otimes \beta).
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If $\alpha$ is a $k$-form and $\beta$ is an $l$-form on $M$, their wedge product $\alpha \wedge \beta$ is the $(k + l)$-form on $M$ defined by

$$\alpha \wedge \beta = \frac{(k + l)!}{k! \ l!} \mathcal{A}(\alpha \otimes \beta).$$

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Tensor and Wedge Products

- If $\alpha$ is a $k$-form and $\beta$ is an $l$-form on $M$, their **wedge product** $\alpha \wedge \beta$ is the $(k + l)$-form on $M$ defined by
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- **Examples:** if $\alpha$ and $\beta$ are one-forms, then
  \[
  (\alpha \wedge \beta)(v_1, v_2) = \alpha(v_1)\beta(v_2) - \alpha(v_2)\beta(v_1),
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If $\alpha$ is a 2-form and $\beta$ is a 1-form,

$$(\alpha \wedge \beta)(v_1, v_2, v_3) = \alpha(v_1, v_2)\beta(v_3) - \alpha(v_1, v_3)\beta(v_2) + \alpha(v_2, v_3)\beta(v_1).$$
Wedge product properties:

(i) **Associative:** $\alpha \wedge (\beta \wedge \gamma) = (\alpha \wedge \beta) \wedge \gamma$.

(ii) **Bilinear:**

$$ (a\alpha_1 + b\alpha_2) \wedge \beta = a(\alpha_1 \wedge \beta) + b(\alpha_2 \wedge \beta), $$
$$ \alpha \wedge (c\beta_1 + d\beta_2) = c(\alpha \wedge \beta_1) + d(\alpha \wedge \beta_2). $$

(iii) **Anticommutative:** $\alpha \wedge \beta = (-1)^{kl} \beta \wedge \alpha$, where $\alpha$ is a $k$-form and $\beta$ is an $l$-form.
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Coordinate Representation: Use dual basis \( dx^i; \)

\( \alpha = \alpha_{i_1...i_k} dx^{i_1} \wedge \cdots \wedge dx^{i_k}, \)

where the sum is over all \( i_j \) satisfying \( i_1 < \cdots < i_k. \)
$\varphi : M \to N$, a smooth map and $\alpha$ a $k$-form on $N$. 

Pull-Back and Push-Forward

- \( \varphi : M \rightarrow N \), a smooth map and \( \alpha \) a \( k \)-form on \( N \).
- **Pull-back:** \( \varphi^* \alpha \) of \( \alpha \) by \( \varphi \): the \( k \)-form on \( M \)

\[
(\varphi^* \alpha)_m(v_1, \ldots, v_k) = \alpha_{\varphi(m)}(T_m \varphi \cdot v_1, \ldots, T_m \varphi \cdot v_k).
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- The pull-back of a wedge product is the wedge product of the pull-backs:

  $$\varphi^*(\alpha \wedge \beta) = \varphi^*\alpha \wedge \varphi^*\beta.$$
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**Product Rule-Like Property.** Let $\alpha$ be a $k$-form and $\beta$ a 1-form on a manifold $M$. Then

$$i_X(\alpha \wedge \beta) = (i_X\alpha) \wedge \beta + (-1)^k \alpha \wedge (i_X\beta).$$

or, in the hook notation,

$$X \hook(\alpha \wedge \beta) = (X \hook \alpha) \wedge \beta + (-1)^k \alpha \wedge (X \hook \beta).$$
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- $d^2 = 0$, that is, $d(d\alpha) = 0$ for any $k$-form $\alpha$.
- $d$ is a **local operator**, that is, $d\alpha(m)$ depends only on $\alpha$ restricted to any open neighborhood of $m$; that is, if $U$ is open in $M$, then
  \[
d(\alpha|U) = (d\alpha)|U.
  \]
If \( \alpha \) is a \( k \)-form given in coordinates by

\[
\alpha = \alpha_{i_1 \ldots i_k} dx^{i_1} \wedge \cdots \wedge dx^{i_k} \quad (\text{sum on } i_1 < \cdots < i_k),
\]

then the coordinate expression for the exterior derivative is

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d\alpha = \frac{\partial \alpha_{i_1 \ldots i_k}}{\partial x^j} dx^j \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_k}.
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This formula is easy to remember from the properties.
Exterior Derivative

Properties.

- Exterior differentiation commutes with pull-back, that is,
  \[ d(\varphi^*\alpha) = \varphi^*(d\alpha), \]
  where \( \alpha \) is a \( k \)-form on a manifold \( N \) and \( \varphi : M \to N \).
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- A \( k \)-form \( \alpha \) is called **closed** if \( d\alpha = 0 \) and is **exact** if there is a \((k-1)\)-form \( \beta \) such that \( \alpha = d\beta \).
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- **Poincaré Lemma** A closed form is locally exact; that is, if \( d\alpha = 0 \), there is a neighborhood about each point on which \( \alpha = d\beta \).
□ Sharp and Flat (Using standard coordinates in $\mathbb{R}^3$)

(a) $v^\flat = v^1 dx + v^2 dy + v^3 dz$, the one-form corresponding to the vector $v = v^1 e_1 + v^2 e_2 + v^3 e_3$.

(b) $\alpha^\sharp = \alpha_1 e_1 + \alpha_2 e_2 + \alpha_3 e_3$, the vector corresponding to the one-form $\alpha = \alpha_1 dx + \alpha_2 dy + \alpha_3 dz$. 

**Sharp and Flat** *(Using standard coordinates in \( \mathbb{R}^3 \))*

(a) \( v^♭ = v^1 \, dx + v^2 \, dy + v^3 \, dz \), the one-form corresponding to the vector \( v = v^1 \mathbf{e}_1 + v^2 \mathbf{e}_2 + v^3 \mathbf{e}_3 \).

(b) \( \alpha^♯ = \alpha_1 \mathbf{e}_1 + \alpha_2 \mathbf{e}_2 + \alpha_3 \mathbf{e}_3 \), the vector corresponding to the one-form \( \alpha = \alpha_1 \, dx + \alpha_2 \, dy + \alpha_3 \, dz \).

**Hodge Star Operator**

(a) \( *1 = dx \wedge dy \wedge dz \).

(b) \( *dx = dy \wedge dz \), \( *dy = -dx \wedge dz \), \( *dz = dx \wedge dy \),

\( *(dy \wedge dz) = dx \), \( *(dx \wedge dz) = -dy \), \( *(dx \wedge dy) = dz \).

(c) \( *(dx \wedge dy \wedge dz) = 1 \).
**Sharp and Flat** (Using standard coordinates in $\mathbb{R}^3$)

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(b) $\alpha^\# = \alpha_1 e_1 + \alpha_2 e_2 + \alpha_3 e_3$, the vector corresponding to the one-form $\alpha = \alpha_1 \, dx + \alpha_2 \, dy + \alpha_3 \, dz$.

**Hodge Star Operator**

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$$\star(dx \wedge dz) = dx, \quad \star(dx \wedge dy) = -dy, \quad \star(dx \wedge dy) = dz.$$  
(c) $* (dx \wedge dy \wedge dz) = 1$.

**Cross Product and Dot Product**

(a) $v \times w = [\star(v^b \wedge w^b)]^\#$.

(b) $(v \cdot w) dx \wedge dy \wedge dz = v^b \wedge \star(w^b)$.
Gradient \n\[ \nabla f = \text{grad } f = (df)^\#. \n\]
\[ \nabla f = \text{grad } f = (\text{d} f)^\#. \]

\[ \nabla \times F = \text{curl } F = [\ast (\text{d} F^b)]^\#. \]
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\[ \nabla \times F = \text{curl } F = [\ast (dF^b)]^\#. \]

\[ \nabla \cdot F = \text{div } F = \ast d(\ast F^b). \]
Dynamic definition: Let $\alpha$ be a $k$-form and $X$ be a vector field with flow $\varphi_t$. The \textit{Lie derivative} of $\alpha$ along $X$ is

$$\mathcal{L}_X \alpha = \lim_{t \to 0} \frac{1}{t} [(\varphi_t^* \alpha) - \alpha] = \frac{d}{dt} \varphi_t^* \alpha \bigg|_{t=0}.$$
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**Extend to non-zero values of $t$:**

$$\frac{d}{dt}\varphi_t^* \alpha = \varphi_t^* \mathcal{L}_X \alpha.$$
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**Extend to non-zero values of $t$:**

$$\frac{d}{dt} \varphi_t^* \alpha = \varphi_t^* \mathcal{L}_X \alpha.$$

**Time-dependent vector fields**

$$\frac{d}{dt} \varphi_{t,s}^* \alpha = \varphi_{t,s}^* \mathcal{L}_X \alpha.$$
Real Valued Functions. The Lie derivative of $f$ along $X$ is the directional derivative

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Real Valued Functions. The *Lie derivative of $f$ along $X$* is the *directional derivative*

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Operator notation: $X[f] = df \cdot X$

The operator is a *derivation*; that is, the product rule holds.
□ **Pull-back.** If $Y$ is a vector field on a manifold $N$ and $\varphi : M \to N$ is a diffeomorphism, the pull-back $\varphi^*Y$ is a vector field on $M$ defined by

$$(\varphi^*Y)(m) = (T_m\varphi^{-1} \circ Y \circ \varphi)(m).$$
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Push-forward. For a diffeomorphism \( \varphi \), the push-forward is defined, as for forms, by \( \varphi_* = (\varphi^{-1})^* \).
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Push-forward. For a diffeomorphism $\varphi$, the **push-forward** is defined, as for forms, by $\varphi_* = (\varphi^{-1})^*$. Flows of $X$ and $\varphi_*X$ related by conjugation.
Lie Derivative

\[ c = \text{integral curve of } X \]

\[ \varphi \circ F_t \circ \varphi^{-1} \]

\[ \varphi \cdot \varphi \circ c \cdot \varphi^{-1} \text{ curve of } \varphi_\ast X \]
The Lie derivative on functions is a \textit{derivation}; conversely, derivations determine vector fields.
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which determines the unique vector field \([X, Y]\) the \textit{Jacobi–Lie bracket} of \(X\) and \(Y\).
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\[ \mathcal{L}_X Y = [X, Y], \quad \text{\textit{Lie derivative}} \quad \text{of} \quad Y \quad \text{along} \quad X. \]
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$\mathcal{L}_X Y = [X, Y]$, Lie derivative of $Y$ along $X$.

The analog of the Lie derivative formula holds.

Coordinates:

$$(\mathcal{L}_X Y)^j = X^i \frac{\partial Y^j}{\partial x^i} - Y^i \frac{\partial X^j}{\partial x^i} = (X \cdot \nabla)Y^j - (Y \cdot \nabla)X^j,$$
The formula for \([X, Y] = \mathcal{L}_XY\) can be remembered by writing

\[
\left[ X^i \frac{\partial}{\partial x^i}, Y^j \frac{\partial}{\partial x^j} \right] = X^i \frac{\partial Y^j}{\partial x^i} \frac{\partial}{\partial x^j} - Y^j \frac{\partial X^i}{\partial x^j} \frac{\partial}{\partial x^i}.
\]
Program: Extend the definition of the Lie derivative from functions and vector fields to differential forms, by requiring that the Lie derivative be a derivation.
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Example. For a 1-form $\alpha$,

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More generally, determine $\mathcal{L}_X \alpha$ by

$$\mathcal{L}_X (\alpha(Y_1, \ldots, Y_k))$$

$$= (\mathcal{L}_X \alpha)(Y_1, \ldots, Y_k) + \sum_{i=1}^{k} \alpha(Y_1, \ldots, \mathcal{L}_X Y_i, \ldots, Y_k).$$
The dynamic and algebraic definitions of the Lie derivative of a differential $k$-form are equivalent.
Equivalence

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Equivalence

- The dynamic and algebraic definitions of the Lie derivative of a differential $k$-form are equivalent.
- The Lie derivative formalism holds for all tensors, not just differential forms.
- Very useful in all areas of mechanics: eg, the rate of strain tensor in elasticity is a Lie derivative and the vorticity advection equation in fluid dynamics are both Lie derivative equations.
Cartan’s Magic Formula. For $X$ a vector field and $\alpha$ a $k$-form

$$\mathcal{L}_X \alpha = d\iota_X \alpha + \iota_X d\alpha,$$
Properties

- **Cartan’s Magic Formula.** For $X$ a vector field and $\alpha$ a $k$-form

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$$\mathcal{L}_X \alpha = d(X \hook \alpha) + X \hook d\alpha.$$
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$$\mathcal{L}_X\alpha = \text{di}_X\alpha + \text{i}_X\text{d}\alpha,$$

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If $\varphi : M \to N$ is a diffeomorphism, then

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for $Y \in \mathfrak{X}(N)$ and $\beta \in \Omega^k(M)$.
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□ **Cartan’s Magic Formula.** For $X$ a vector field and $\alpha$ a $k$-form

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for $Y \in \mathfrak{X}(N)$ and $\beta \in \Omega^k(M)$.

□ Many other useful identities, such as

\[ d\Theta(X, Y) = X[\Theta(Y)] - Y[\Theta(X)] - \Theta([X, Y]). \]
An $n$-manifold $M$ is *orientable* if there is a nowhere-vanishing $n$-form $\mu$ on it; $\mu$ is a *volume form*.
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**Oriented Basis.** A basis $\{v_1, \ldots, v_n\}$ of $T_m M$ is **positively oriented** relative to the volume form $\mu$ on $M$ if $\mu(m)(v_1, \ldots, v_n) > 0$. 

**Volume Forms and Divergence**

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- **Oriented Basis.** A basis $\{v_1, \ldots, v_n\}$ of $T_mM$ is **positively oriented** relative to the volume form $\mu$ on $M$ if $\mu(m)(v_1, \ldots, v_n) > 0$.

- **Divergence.** If $\mu$ is a volume form, there is a function, called the **divergence** of $X$ relative to $\mu$ and denoted by $\text{div}_\mu(X)$ or simply $\text{div}(X)$, such that

$$\mathcal{L}_X \mu = \text{div}_\mu(X) \mu.$$
Dynamic approach to Lie derivatives $\Rightarrow \text{div}_\mu(X) = 0$ if and only if $F_t^*\mu = \mu$, where $F_t$ is the flow of $X$ (that is, $F_t$ is \textit{volume preserving}.)}
Dynamic approach to Lie derivatives $\Rightarrow \text{div}_\mu(X) = 0$ if and only if $F_t^* \mu = \mu$, where $F_t$ is the flow of $X$ (that is, $F_t$ is *volume preserving*.)

If $\varphi : M \to M$, there is a function, called the *Jacobian* of $\varphi$ and denoted by $J_\mu(\varphi)$ or simply $J(\varphi)$, such that

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Dynamic approach to Lie derivatives $\Rightarrow \text{div}_\mu(X) = 0$ if and only if $F_t^*\mu = \mu$, where $F_t$ is the flow of $X$ (that is, $F_t$ is \textit{volume preserving}.)

If $\varphi : M \rightarrow M$, there is a function, called the \textbf{Jacobian} of $\varphi$ and denoted by $J_\mu(\varphi)$ or simply $J(\varphi)$, such that 

$$\varphi^*\mu = J_\mu(\varphi)\mu.$$ 

Consequence: $\varphi$ is \textit{volume preserving} if and only if $J_\mu(\varphi) = 1$. 
A vector subbundle (a regular distribution) $E \subset TM$ is involutive if for any two vector fields $X, Y$ on $M$ with values in $E$, the Jacobi–Lie bracket $[X, Y]$ is also a vector field with values in $E$. 
Frobenius’ Theorem

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- $E$ is **integrable** if for each $m \in M$ there is a local submanifold of $M$ containing $m$ such that its tangent bundle equals $E$ restricted to this submanifold.
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- **Frobenius theorem**: $E$ is involutive if and only if it is integrable.
Stokes’ Theorem

**Idea:** Integral of an $n$-form $\mu$ on an oriented $n$-manifold $M$: pick a covering by coordinate charts and sum up the ordinary integrals of $f(x^1, \ldots, x^n) \, dx^1 \cdots dx^n$, where

$$
\mu = f(x^1, \ldots, x^n) \, dx^1 \wedge \cdots \wedge dx^n
$$

(don’t count overlaps twice).
**Stokes’ Theorem**

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  (don’t count overlaps twice).

- The change of variables formula guarantees that the result, denoted by $\int_M \mu$, is well-defined.

- **Oriented manifold with boundary:** the boundary, $\partial M$, inherits a compatible orientation: generalizes the relation between the orientation of a surface and its boundary in the classical Stokes’ theorem in $\mathbb{R}^3$. 


Stokes’ Theorem
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Suppose that $M$ is a compact, oriented $k$-dimensional manifold with boundary $\partial M$. Let $\alpha$ be a smooth $(k - 1)$-form on $M$. Then

$$\int_M d\alpha = \int_{\partial M} \alpha.$$
Stokes’ Theorem

**Stokes’ Theorem** Suppose that $M$ is a compact, oriented $k$-dimensional manifold with boundary $\partial M$. Let $\alpha$ be a smooth $(k-1)$-form on $M$. Then

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**Special cases:** The classical vector calculus theorems of Green, Gauss and Stokes.
Stokes’ Theorem

(a) **Fundamental Theorem of Calculus.**
\[ \int_a^b f'(x) \, dx = f(b) - f(a). \]

(b) **Green’s Theorem.** For a region \( \Omega \subset \mathbb{R}^2 \),
\[ \iint_{\Omega} \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \, dx \, dy = \oint_{\partial \Omega} P \, dx + Q \, dy. \]

(c) **Divergence Theorem.** For a region \( \Omega \subset \mathbb{R}^3 \),
\[ \iiint_{\Omega} \text{div} \, \mathbf{F} \, dV = \iint_{\partial \Omega} \mathbf{F} \cdot \mathbf{n} \, dA. \]
(d) Classical Stokes’ Theorem. For a surface $S \subset \mathbb{R}^3$, \[
\int \int_S \left\{ \left( \frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} \right) dy \wedge dz \\
+ \left( \frac{\partial P}{\partial z} - \frac{\partial R}{\partial x} \right) dz \wedge dx + \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx \wedge dy \right\} \\
= \int \int_S \mathbf{n} \cdot \text{curl} \mathbf{F} \, dA = \int_{\partial S} P \, dx + Q \, dy + R \, dz,
\]
where $\mathbf{F} = (P, Q, R)$. 
Stokes’ Theorem

- **Poincaré lemma:** generalizes vector calculus theorems: if $\text{curl } \mathbf{F} = 0$, then $\mathbf{F} = \nabla f$, and if $\text{div } \mathbf{F} = 0$, then $\mathbf{F} = \nabla \times \mathbf{G}$. 
\[ \text{Poincaré lemma:} \] generalizes vector calculus theorems: if \( \text{curl} \, \mathbf{F} = 0 \), then \( \mathbf{F} = \nabla f \), and if \( \text{div} \, \mathbf{F} = 0 \), then \( \mathbf{F} = \nabla \times \mathbf{G} \).

\[ \text{Recall: if } \alpha \text{ is closed, then locally } \alpha \text{ is exact; that is, if } \text{d} \alpha = 0, \text{ then locally } \alpha = \text{d} \beta \text{ for some } \beta. \]
Poincaré lemma: generalizes vector calculus theorems: if \( \text{curl } \mathbf{F} = 0 \), then \( \mathbf{F} = \nabla f \), and if \( \text{div } \mathbf{F} = 0 \), then \( \mathbf{F} = \nabla \times \mathbf{G} \).

Recall: if \( \alpha \) is closed, then locally \( \alpha \) is exact; that is, if \( d\alpha = 0 \), then locally \( \alpha = d\beta \) for some \( \beta \).

Calculus Examples: need not hold globally:

\[
\alpha = \frac{xdy - ydx}{x^2 + y^2}
\]

is closed (or as a vector field, has zero curl) but is not exact (not the gradient of any function on \( \mathbb{R}^2 \) minus the origin).
\( M \) and \( N \) oriented \( n \)-manifolds; \( \varphi : M \to N \) an orientation-preserving diffeomorphism, \( \alpha \) an \( n \)-form on \( N \) (with, say, compact support), then

\[
\int_M \varphi^* \alpha = \int_N \alpha.
\]
Identities for Vector Fields and Forms

- Vector fields on $M$ with the bracket $[X, Y]$ form a **Lie algebra**; that is, $[X, Y]$ is real bilinear, skew-symmetric, and **Jacobi’s identity** holds:

$$[[X, Y], Z] + [[Z, X], Y] + [[Y, Z], X] = 0.$$  

Locally,

$$[X, Y] = (X \cdot \nabla)Y - (Y \cdot \nabla)X,$$

and on functions,

$$[X, Y][f] = X[Y[f]] - Y[X[f]].$$

- For diffeomorphisms $\varphi$ and $\psi$,

$$\varphi_*[X, Y] = [\varphi_*X, \varphi_*Y] \quad \text{and} \quad (\varphi \circ \psi)_*X = \varphi_*\psi_*X.$$

- $(\alpha \wedge \beta) \wedge \gamma = \alpha \wedge (\beta \wedge \gamma)$ and $\alpha \wedge \beta = (-1)^{kl} \beta \wedge \alpha$ for $k$- and $l$-forms $\alpha$ and $\beta$.

- For maps $\varphi$ and $\psi$,

$$\varphi^*(\alpha \wedge \beta) = \varphi^*\alpha \wedge \varphi^*\beta \quad \text{and} \quad (\varphi \circ \psi)^*\alpha = \psi^*\varphi^*\alpha.$$
Identities for Vector Fields and Forms

- $d$ is a real linear map on forms, $dd\alpha = 0$, and
  $$d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^k \alpha \wedge d\beta$$
  for $\alpha$ a $k$-form.

- For $\alpha$ a $k$-form and $X_0, \ldots, X_k$ vector fields,
  $$d\alpha(X_0, \ldots, X_k) = \sum_{i=0}^{k} (-1)^i X_i[\alpha(X_0, \ldots, \hat{X}_i, \ldots, X_k)]$$
  $$+ \sum_{0 \leq i < j \leq k} (-1)^{i+j} \alpha([X_i, X_j], X_0, \ldots, \hat{X}_i, \ldots, \hat{X}_j, \ldots, X_k),$$
  where $\hat{X}_i$ means that $X_i$ is omitted. Locally,
  $$d\alpha(x)(v_0, \ldots, v_k) = \sum_{i=0}^{k} (-1)^i D\alpha(x) \cdot v_i(v_0, \ldots, \hat{v}_i, \ldots, v_k).$$

- For a map $\varphi$,
  $$\varphi^* d\alpha = d\varphi^* \alpha.$$
Identities for Vector Fields and Forms

○ **Poincaré Lemma.** If $d\alpha = 0$, then the $k$-form $\alpha$ is locally exact; that is, there is a neighborhood $U$ about each point on which $\alpha = d\beta$. This statement is global on contractible manifolds or more generally if $H^k(M) = 0$.

○ $i_X\alpha$ is real bilinear in $X$, $\alpha$, and for $h : M \to \mathbb{R}$,

$$i_{hX}\alpha = hi_X\alpha = i_X h\alpha.$$  

Also, $i_Xi_X\alpha = 0$ and

$$i_X(\alpha \wedge \beta) = i_X\alpha \wedge \beta + (-1)^k\alpha \wedge i_X\beta$$

for $\alpha$ a $k$-form.

○ For a diffeomorphism $\varphi$,

$$\varphi^*(i_X\alpha) = i_{\varphi^*X}(\varphi^*\alpha), \quad \text{i.e.,} \quad \varphi^*(X \lrcorner \alpha) = (\varphi^*X) \lrcorner (\varphi^*\alpha).$$

○ If $f : M \to N$ is a mapping and $Y$ is $f$-related to $X$, that is,

$$Tf \circ X = Y \circ f,$$
then
\[ i_X f^* \alpha = f^* i_Y \alpha; \quad \text{i.e.,} \quad X \lrcorner (f^* \alpha) = f^* (Y \lrcorner \alpha). \]

- \( \mathcal{L}_x \alpha \) is real bilinear in \( X, \alpha \) and
  \[ \mathcal{L}_X (\alpha \wedge \beta) = \mathcal{L}_X \alpha \wedge \beta + \alpha \wedge \mathcal{L}_X \beta. \]

- Cartan’s Magic Formula:
  \[ \mathcal{L}_X \alpha = d i_X \alpha + i_X d \alpha = d(X \lrcorner \alpha) + X \lrcorner d \alpha. \]

- For a diffeomorphism \( \varphi \),
  \[ \varphi^* \mathcal{L}_X \alpha = \mathcal{L}_{\varphi^* X} \varphi^* \alpha. \]

If \( f : M \to N \) is a mapping and \( Y \) is \( f \)-related to \( X \), then
\[ \mathcal{L}_Y f^* \alpha = f^* \mathcal{L}_X \alpha. \]
Identities for Vector Fields and Forms

\((\mathcal{L}_X \alpha)(X_1, \ldots, X_k) = X[\alpha(X_1, \ldots, X_k)] - \sum_{i=0}^{k} \alpha(X_1, \ldots, [X, X_i], \ldots, X_k).\)

Locally,

\((\mathcal{L}_X \alpha)(x) \cdot (v_1, \ldots, v_k) = (D \alpha_x \cdot X(x))(v_1, \ldots, v_k) + \sum_{i=0}^{k} \alpha_x(v_1, \ldots, DX_x \cdot v_i, \ldots, v_k).\)

More identities:

- \(\mathcal{L}_f X \alpha = f \mathcal{L}_X \alpha + df \wedge i_X \alpha;\)
- \(\mathcal{L}_{[X,Y]} \alpha = \mathcal{L}_X \mathcal{L}_Y \alpha - \mathcal{L}_Y \mathcal{L}_X \alpha;\)
- \(i_{[X,Y]} \alpha = \mathcal{L}_X i_Y \alpha - i_Y \mathcal{L}_X \alpha;\)
- \(\mathcal{L}_X d\alpha = d\mathcal{L}_X \alpha;\)
- \(\mathcal{L}_X i_X \alpha = i_X \mathcal{L}_X \alpha;\)
• $\mathcal{L}_X(\alpha \wedge \beta) = \mathcal{L}_X\alpha \wedge \beta + \alpha \wedge \mathcal{L}_X\beta$. 
Identities for Vector Fields and Forms

- **Coordinate formulas:** for \( X = X^l \partial / \partial x^l \), and

\[
\alpha = \alpha_{i_1 \ldots i_k} dx^{i_1} \wedge \cdots \wedge dx^{i_k},
\]

where \( i_1 < \cdots < i_k \):

- \( d\alpha = \left( \frac{\partial \alpha_{i_1 \ldots i_k}}{\partial x^l} \right) dx^l \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_k} \),

- \( \mathbf{i}_X \alpha = X^l \alpha_{li_2 \ldots i_k} dx^{i_2} \wedge \cdots \wedge dx^{i_k} \),

- \( \mathcal{L}_X \alpha = X^l \left( \frac{\partial \alpha_{i_1 \ldots i_k}}{\partial x^l} \right) dx^{i_1} \wedge \cdots \wedge dx^{i_k} + \alpha_{li_2 \ldots i_k} \left( \frac{\partial X^l}{\partial x^{i_1}} \right) dx^{i_1} \wedge dx^{i_2} \wedge \cdots \wedge dx^{i_k} + \ldots. \)