CALIFORNIA INSTITUTE OF TECHNOLOGY
Control and Dynamical Systems

CDS140

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Problem Set #4

Issued: 1 Feb 16
Due: 10 Feb 16

1. **Perko, Section 2.9, problem 3** Use the Lyapunov function $V(x) = x_1^2 + x_2^2 + x_3^2$ to show that the origin is an asymptotically stable equilibrium point of the system

$$\dot{x} = \begin{bmatrix} -x_2 - x_1x_2^2 + x_3^2 - x_1^3 \\ x_1 + x_3^3 - x_2^3 \\ -x_1x_3 - x_3x_1^2 - x_2x_3^2 - x_3^5 \end{bmatrix}$$

Show that the trajectories of the linearized system $\dot{x} = Df(0)x$ for this problem lie on circles in planes parallel to the x_1, x_2 plane; hence, the origin is stable, but not asymptotically stable for the linearized system.

2. Determine the stability of the system

$$\begin{aligned} \dot{x} &= -y - x^3 \\ \dot{y} &= x^5 \end{aligned}$$

Hint: motivated by the first equation, try a Lyapunov function of the form

$$V(x, y) = \alpha x^6 + \beta y^2$$

Is the origin asymptotically stable? Is the origin globally asymptotically stable?

3. Definition: An equilibrium point is "exponentially stable" if $\exists\, M$, $\alpha > 0$ and $\epsilon > 0$ such that $\|x(t)\| \le M e^{-\alpha t}\|x(0)\|$, $\forall \|x(0)\| \le \epsilon$, $t \ge 0$.

Let $\dot{x} = f(x)$ be a dynamical system with an equilibrium point at $x_e = 0$. Show that if there is a function $V(x)$ satisfying

$$k_1\|x\|^2 \le V(x) \le k_2\|x\|^2, \quad \dot{V}(x) \le -k_3\|x\|^2$$

for positive constants k_1, k_2 and k_3, then the equilibrium point at the origin is exponentially stable. Compute M and α in terms of k_1, k_2, and k_3.

4. **Perko, Section 2.12, problem 2** Use Theorem 1 [Center Manifold Theorem] to determine the qualitative behaviour near the non-hyperbolic critical point at the origin for the system

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= -y + \alpha x^2 + xy \end{aligned}$$

for $\alpha \neq 0$ and for $\alpha = 0$; i.e., follow the procedure in Example 1 after diagonalizing the system as in Example 3.

5. Consider the following system in $\mathbb{R}^2$:

$$\dot{x} = -\frac{\alpha}{2}(x^2 + y^2) + \alpha(x + y) - \alpha$$
$$\dot{y} = -\alpha xy + \alpha(x + y) - \alpha$$

Determine the stable, unstable, and centre manifold of the equilibrium point at $(x, y) = (1, 1)$, and determine the stability of this equilibrium point for $\alpha \neq 0$. For determining stability, note that near the equilibrium point there are two 1-dimensional invariant linear manifolds of the form $M = \{(a_1, a_2) \in \mathbb{R}^2 \| a_2 = ka_1\}$; determine the flow on these invariant manifolds.