CALIFORNIA INSTITUTE OF TECHNOLOGY
Control and Dynamical Systems

CDS140

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Problem Set #1

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1. **Perko, Section 2.2, problem 5:** Let V be a normed linear space. If $T : V \to V$ satisfies

$$\|T(u) - T(v)\| \leq c\|u - v\|$$

for all $u, v \in V$ with $0 < c < 1$ then T is called a *contraction mapping*. It can be shown that contraction mappings give rise to unique solutions of the equation $T(u) = u$:

Theorem (Contraction Mapping Principle) Let V be a complete normed linear space and $T : V \to V$ a contraction mapping. Then there exists a unique $u \in V$ such that $T(u) = u$.

Let $f \in C^1(E)$ and $x_0 \in E$. For $I = [-a, a]$ and $u \in C(I)$, let

$$T(u)(t) = x_0 + \int_0^t f(u(s))ds.$$

Define a closed subset V of $C(I)$ and apply the Contraction Mapping Principle to show that the integral equation (7) in Perko, Section 2.2 has a unique solution $u(t)$ for all $t \in [-a, a]$ provided the constant $a > 0$ is sufficiently small.

2. **Perko, Section 2.3, problem 1**: Use the fundamental theorem for linear systems in Chapter 1 of Perko to solve the initial value problem

$$\dot{x} = Ax, \qquad x(0) = y.$$

Let $u(t, y)$ denote the solution and compute

$$\Phi(t) = \frac{\partial u}{\partial y}(t, y).$$

Show that $\Phi(t)$ is the fundamental matrix solution of

$$\dot{\Phi} = A\Phi, \qquad \Phi(0) = I.$$

(Note: this problem works through the more general result for nonlinear systems (Corollary on page 83) for the special case of a linear system.)

3. **Perko, Section 2.5, problem 4**: Sketch the flow of the linear system

$$\dot{x} = Ax \quad \text{with} \quad A = \begin{bmatrix} -1 & -3 \\ 0 & 2 \end{bmatrix}$$

and describe $\phi_t(N_\epsilon(x_0))$ for $x_0 = (-3, 0)$, $\epsilon = 0.2$. (A qualitative sketch is sufficient, like Figure 2 in Section 1.2.)

4. **Perko, Section 2.5, problem 5**: Determine the flow $\phi_t : \mathbb{R}^2 \to \mathbb{R}^2$ for the nonlinear system

$$\dot{x} = f(x) \quad \text{with} \quad f(x) = \begin{bmatrix} -x_1 \\ 2x_2 + x_1^2 \end{bmatrix}$$

and show that the set $S = \{x \in \mathbb{R}^2 | x_2 = -x_1^2/4\}$ is invariant with respect to the flow ϕ_t.

5. Choose *one* of the following systems and determine all of the equilibrium points for the system, indicating whether each is a sink, source, or saddle.

(a) Moore-Greitzer model: The Moore-Greitzer equations model rotating stall and surge in gas turbine engines are given by

$$\begin{aligned} \frac{d\psi}{dt} &= \frac{1}{4B^2 l_c} \left(\phi - \Phi_T(\psi)\right), \\ \frac{d\phi}{dt} &= \frac{1}{l_c} \left(\Psi_c(\phi) - \psi + \frac{J}{8} \frac{\partial^2 \Psi_c}{\partial \phi^2}\right), \\ \frac{dJ}{dt} &= \frac{2}{\mu + m} \left(\frac{\partial \Psi_c}{\partial \phi} + \frac{J}{8} \frac{\partial^3 \Psi_c}{\partial \phi^3}\right) J, \end{aligned}$$

where

$$\begin{aligned} B &= 0.2, & \Phi_T(\psi) &= \sqrt{\psi}, \\ l_c &= 6, & \Psi_c(\phi) &= 1 + 1.5\phi - 0.5\phi^3, \\ \mu &= 1.256, & m &= 2. \end{aligned}$$

This is a model for the dynamics of the compression system (first part of a jet engine) with ψ representing the pressure rise across the compressor, ϕ representing the mass flow through the compressor and J representing the amplitude squared of the first modal flow perturbation (corresponding to a rotating stall disturbance).

(b) Genetic toggle switch: Consider the dynamics of two transcriptional repressors connected together in a cycle. It can be shown that the normalized dynamics of the system can be written as

$$\frac{dz_1}{d\tau} = \frac{\mu}{1 + z_2^n} - z_1 - v_1, \qquad \frac{dz_2}{d\tau} = \frac{\mu}{1 + z_1^n} - z_2 - v_2.$$

where z_1 and z_2 represent scaled versions of the protein concentrations, v_1 and v_2 represent external inputs and the time scale has been changed. Let $\mu = 2.16$, $n = 2$ and $v_1 = v_2 = 0$.

(c) Congestion control: A simplified model for congestion control between N computers connected by a router is given by the differential equation

$$\frac{dx_i}{dt} = -b\frac{x_i^2}{2} + (b_{\max} - b), \qquad \frac{db}{dt} = \left(\sum_{i=1}^{N} x_i\right) - c,$$

where $x_i \in \mathbb{R}$, $i = 1, \ldots, N$ are the transmission rates for the sources of data, $b \in \mathbb{R}$ is the current buffer size of the router, $b_{\max} > 0$ is the maximum buffer size and $c > 0$ is the capacity of the link connecting the router to the computers. The $\dot{x}_i$ equation represents the control law that the individual computers use to determine how fast to send data across the network and the $\dot{b}$ equation represents the rate at which the buffer on the router fills up. Consider the case where $N = 2$ (so that we have three states, x_1, x_2 and b) and take $b_{\max} = 1$ Mb and $c = 2$ Mb/s.