Linear Differential Equations

Linear system of differential equations:

$$\dot{x} = Ax$$

where $x \in R^n$, A is an $n \times n$ matrix and $\dot{x}$ has the form

$$\dot{x} = \frac{dx}{dt} = \begin{bmatrix} \frac{dx_1}{dt} \\ \vdots \\ \frac{dx_n}{dt} \end{bmatrix}$$

The solution of the linear system, $\dot{x} = Ax$, with initial condition $x(0) = x_0$ is:

$$x(t) = e^{At}x_0$$

where e^{At} is an $n \times n$ matrix function defined by its Taylor series

Uncoupled Systems

Separation of variables can be used to solve the scalar first-order linear differential equation

$$\dot{x} = ax$$

The general solution is given by

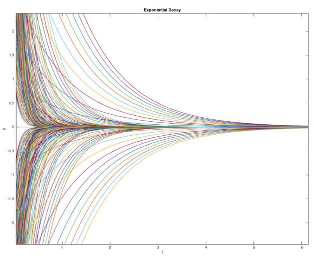
$$x(t) = ce^{at}$$

where the constant $c = x(t@0)$

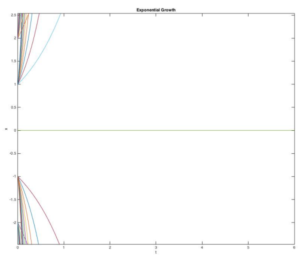
The assumption that a is a constant is referred to as Malthus's law corresponding to Malthus's equation

$$\dot{x} = ax$$

The figure below displays the family of solutions of differential equations for various initial values x_0 with $a < 0$ and $a > 0$. Each initial condition x_0 determines a curve. This image displays the **flow** of the differential equation. The flow, $\phi(t, x_0)$ of an autonomous differential equation is the function of time t and initial value x_0 which represents the set of solutions. Thus $\phi(t, x_0)$ is the value at time t of the solution with initial value x_0



(a) $a < 0$ exponential decay



(b) $a > 0$ exponential growth

Example: Consider the uncoupled linear system

$$\dot{x_1} = -3x_1$$

$$\dot{x_2} = x_2$$

Written in matrix form

$$\dot{x} = Ax$$

where

$$A = \begin{bmatrix} -3 & 0 \\ 0 & 1 \end{bmatrix}$$

The solution to the differential equation again can be found by the separation of variable:

$$x_1 = c_1 e^{-3t}$$

$$x_2 = c_2 e^t$$

or

$$x(t) = \begin{bmatrix} e^{-3t} & 0 \\ 0 & e^t \end{bmatrix} c$$

Each initial condition $x_0 \in R^2$

The figure below displays the set of solution curves for in the x_1-x_2 known as a phase portrait

The phase portrait of a system of differential equations such as

$$\dot{x} = Ax$$

with $x \in R^n$is the set of all solution curves of

$$\dot{x} = Ax$$

in the phase space R^n

The arrows indicate the direction of motion as $t \to \infty$ or $t \to -\infty$

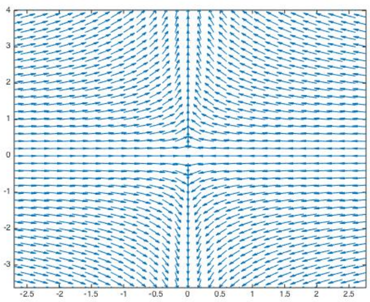
For the example provided the flow ϕ is the mapping $R \times R^2 \to R^2$

Additionally, the function

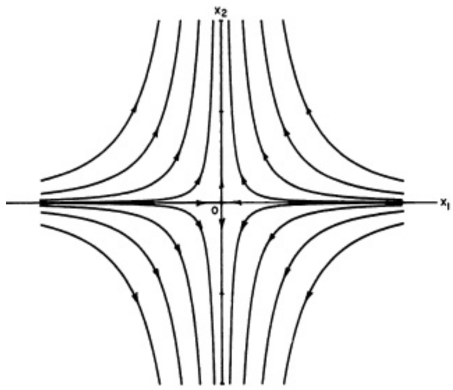
$$f(x) = Ax$$

defines a vector field on R^2

Note that at each point x in the phase space R^2, the solution curves are tangent to the vectors in the vector field Ax. This follows since at time $t = t_0$, $\dot{x} = (t_0)$ is tangent to the curve $x = x(t)$ at the point $x_0 = x(t_0)$ and since $\dot{x} = Ax$ along the solution curves This is proven in the next section of notes.



(a) Vector field of example



(b) Phase Portrait

Matrix exponential

Given $A \in M(n, F)$, $F = \mathbb{R}$ or $\mathbb{C}$, we define e^A by the power series:

$$e^A = \sum_{k=0}^{\infty} \frac{1}{k!} A^k$$

Recall: $\| A^k \| \leq \| A \|^k$

Hence following the Weierstrass M-test outlined in section 1.3 of Perko

$$e^{At} = \sum_{k=0}^{\infty} \frac{t^k}{k!} A^k$$

is shown to be absolutely convergent for each $A \in M(n, F)$ for all $t \in R$

The fundamental theorem of linear systems

For the initial value problem

$$\dot{x} = Ax$$

$$x(0) = x_0$$

there is a unique solution given by

$$x(t) = e^{At}x_0$$

Proof: Outlined in section 1.4 of Perko

Proposition: Given $A \in M(n, \mathbb{C})$, $s, t \in R$,

$$e^{(s+t)A} = e^{sA}e^{tA}$$

For $A, B \in M(n, F)$ where A and B commute then

$$e^A e^B = e^{A+B}$$

Proofs: Outlined in section 1.3 of Perko

Diagonalization

Diagonalization can be used to reduce a linear system

$$\dot{x} = Ax$$

to an uncoupled system, where A is a square matrix and eigenvalues $\lambda_l, \lambda_2, ..., \lambda_n$ of a $n \times n$ matrix A are real and distinct

Then any set of corresponding eigenvectors $[v_l, v_2, ..., v_n]$ form a basis for R^n, the matrix $P = [v_1, v_2, ..., v_n]$ is invertible and $P^{-1}AP = \text{diag}[\lambda_1, ..., \lambda_n]$

Therefore if a linear transformation $T : R^n \to R^n$ is represented by the $n \times n$ matrix A with respect to the standard basis $[e_1, e_2, ..., e_n]$ for R^n,then with respect to any basis of eigenvectors $[v_1, v_2, ..., v_n]$

T is represented by the diagonal matrix of eigenvalues, $\text{diag}[\lambda_l, \lambda_2, ..., \lambda_n]$

In order to reduce the system, $\dot{x} = Ax$ to an uncoupled linear system using the above theorem, define the linear transformation of coordinates

$$y = P^{-1}x$$

where P is the invertible matrix defined in the theorem

Then

$$x = Py$$

$\dot{y} = P^{-1}\dot{x} = P^{-1}Ax = P^{-1}APy$ and from there we obtain the uncoupled linear system

$$\dot{y} = \text{diag}[\lambda_1, ..., \lambda_n]y$$

corresponding to the solution

$$y(t) = \text{diag}[e^{\lambda_1 t}, ..., e^{\lambda_n t}]y(0)$$

And then since $y(0) = P^{-1}x(0)$ and $x(t) = Py(t)$, it follows that $\dot{x} = Ax$ has the solution

$$x(t) = PE(t)P^{-1}x(0)$$

where $E(t)$is the diagonal matrix

$$E(t) = \text{diag}[e^{\lambda_1 t}, ..., e^{\lambda_n t}]$$

Stable, Unstable, and Center Subspace

Let E^s, E^u, and E^c denote the stable, unstable, and center subspace respectively
Let $w_j = u_j + iv_j$ be a generalized eigenvector of the real matrix A corresponding to an eigenvalue $\lambda_j = a_j + ib_j$

Then

$$E^s = \text{Span}\{u_j, v_j | a_j < 0\}$$
$$E^u = \text{Span}\{u_j, v_j | a_j > 0\}$$
$$E^c = \text{Span}\{u_j, v_j | a_j = 0\}$$

The interesting stuff happens on the center subspace

Non-homogeneous Linear System

The non-homogeneous linear system is of the form

$$\dot{x} = Ax + b(t)$$

where A is a $n \times n$ matrix and $b(t)$ is a continuous vector valued function The solution of the non-homogeneous linear system has the form:

$$x(t) = e^{At}x_0 + e^{At}\int_0^t e^{-A\tau}b(\tau)d\tau$$