Outline: 1) Prove Lyapunov
2) Extend to global stability
3) La Salle's Thm (allow asy. stab. w/ $\dot{V} \le 0$ in special cases)

Proof

a) Show $\dot{V}(x) \le 0 \ \forall x \in E \Rightarrow \forall \epsilon > 0, \exists \delta > 0$ s.t. $\forall x \in N_\delta(0)$ & $t \ge 0$, $\phi_t(x) \in N_\epsilon(0)$ } stable

assume eq. pt. $x_0 = 0$

c) Show $\dot{V}(x) > 0 \ \forall x \in E \setminus \{x_0\}$

$\Rightarrow$ unstable (The above not True)

b) Show $\dot{V}(x) < 0 \ \forall x \in E \setminus \{x_0\}$

$\Rightarrow$ stable and $\phi_t(x) \to 0$ as $t \to \infty$

1. For any $\epsilon > 0$, choose $0 < r \le \epsilon$
so That $B_r = \{x \in \mathbb{R}^n \mid \|x\| \le r\} \subset E$ (closed ball)

(Technicality in case $B_\epsilon \not\subset E$) (i.e. ensure we remain in The domain)

2. a) Let $\alpha = \min_{|x|=r} V(x)$ = ~~maximum~~ value of V on boundary of B_r

b) Take $0 < \beta < \alpha$

Define $\Omega_\beta = \{x \in B_r \mid V(x) \le \beta\}$

So Ω_β is in interior of B_r. (For any a on boundary, $V(a) \ge \alpha > \beta$)

$\forall t \ge 0, \forall x \in E$

c) $V(\phi_t(x)) - V(\phi_0(x))$

$= \int_0^t \frac{d}{ds} V(\phi_s(x))\,ds$

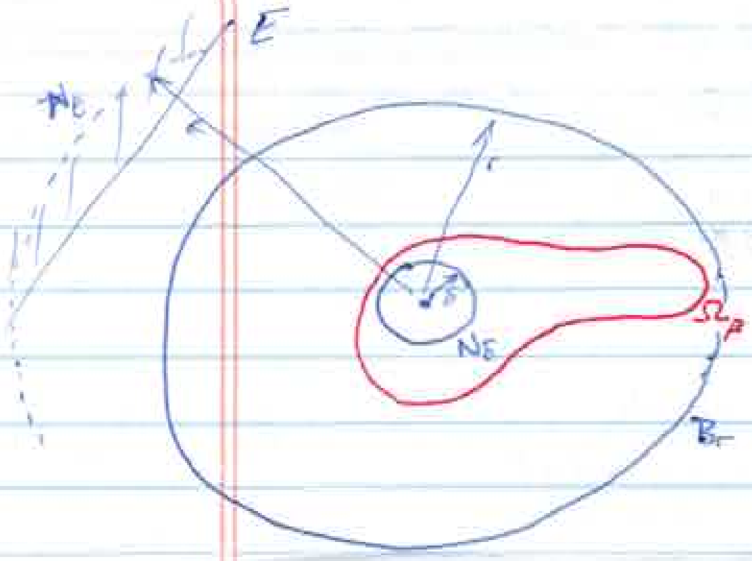
≤ 0 if $\dot{V}(x) \le 0 \ \forall x \in E$

So if $x \in \Omega_\beta$, $V(\phi_t(x)) \le V(\phi_0(x)) \le \beta < \alpha$

If x starts in Ω_β, it stays in Ω_β (positively invariant set)

3. Since V is continuous and $V(0) = 0$, $\exists \delta > 0$ s.t. $\|x\| < \delta \Rightarrow V(x) < \beta$

Thus $x \in N_\delta(0) \Rightarrow x \in \Omega_\beta \Rightarrow \phi_t(x) \in \Omega_\beta \Rightarrow \phi_t(x) \in B_r \Rightarrow \phi_t(x)$
[Hence stable]



b) [Asy. stable if $\dot V < 0$]   Need to show $\exists \delta$ s.t. $\forall x \in N_\delta(0)$ $\lim_{t\to\infty} \varphi_t(x) = 0$

or equivalently, $\lim_{t\to\infty} V(\varphi_t(x)) = 0$

Already proved stability   (since $V=0$ only @ $x=0$)

Since V is a decreasing fn and bounded below,

$\lim_{t\to\infty} V(\varphi_t(x)) = c \geq 0$ ← if not, could have $V \to -\infty$

Suppose $c > 0$, and define $\Omega_c = \{x \in B_r \mid V(x) \leq c\}$

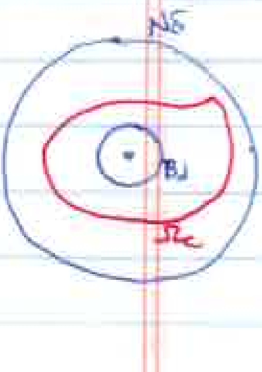
Since V is continuous and $V(0) = 0$, $\exists d > 0$ st

$B_d = \{x \in \mathbb{R}^n \mid \|x\| \leq d\} \subset \Omega_c$, and $V(x) < c \;\; \forall x \in B_d$

Since $\lim_{t\to\infty} V(\varphi_t(x)) = c$, $\varphi_t(x)$ is outside B_d; $d \leq \|\varphi_t(x)\| \leq r \quad \forall t \geq 0$ $\forall x \in N_\delta(0)$

But $\dot V < 0 \;\; \forall x \neq 0$, so $\max_{d \leq |x| \leq r} \dot V(x) = \alpha < 0$ slowest descent

for some α



$$V(\varphi_t(x)) = V(\varphi_0(x)) + \int_0^t \frac{d}{ds} V(\varphi_s(x))\,ds$$

$$\leq V(\varphi_0(x)) + \alpha t$$

$\Rightarrow \exists T$ st $V(\varphi_T(x)) < 0$ $\Rightarrow$ contradiction

(That is, $\lim_{t\to\infty} \dot V = 0$, hence...)

c) [$\dot{V} > 0 \ \forall x \in E \Rightarrow$ unstable]

Reverse Time $(t \to -t)$, get part (b)

Remarks

1. V satisfying conditions is a Lyapunov Function
2. Can determine stability w/o explicitly solving diff. eqn.
3. Doesn't say how to find V...
   - Try quadratic, $V = x^T P x$ (guaranteed to work for linear systems: $\dot{V} = x^T (A^T P + PA) x$
     < 0 if $\underbrace{A^T P + PA + Q = 0}_{\text{Lyapunov equation}}$ for $Q > 0$
   - Simpler still:
     $V(x) = \alpha_1 x_1^2 + \alpha_2 x_2^2 + \dots$
   - Try quartic...
   - Try "Sum of Squares" $\Rightarrow$ SOS Toolbox
4. The region of attraction of an eq. pt. at the origin is $\left\{ x \in \mathbb{R}^n \,\middle|\, \lim_{t \to \infty} \phi_t(x) = 0 \right\}$

Note $\Omega_\beta \subseteq$ RoA... can estimate RoA by maximizing β

Global stability in a minute...

Examples

1. $\dot{x} = -x + y + xy$
   $\dot{y} = x - y - x^2 - y^3$

$\left(\det \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} = 0 \text{ , so } \lambda = 0, -2 \right)$

Try $V = x^2 + y^2$ ... $\dot{V} = 2x(-x + y + xy) + 2y(x - y - x^2 - y^3)$

$V(x,y) > 0 \ \forall (x,y) \neq (0,0)$
$V(0,0) = 0$

$$= -2x^2 + 4xy - 2y^2 + \cancel{2x^2y} - \cancel{2x^2y} - 2y^4$$
$$= -(x-y)^2 - 2y^4$$
$$< 0 \quad \forall x, y \qquad \Rightarrow \text{asy. stable}$$

2. $\ddot{x} + kx = 0, \ k > 0$ (oscillator)
   $\dot{x} = y$
   $\dot{y} = -kx$

(eig. @ $\pm i\sqrt{k}$)

Energy $= \frac{1}{2}kx^2 + \frac{1}{2}y^2$ (kinetic + potential)

(Again, check $V(0,0) = 0$, $V(x,y) > 0$ for $(x,y) \in \mathbb{R}^2 \setminus (0,0)$

$$\dot{V} = kx(y) + y(-kx)$$
$$= 0 \qquad \Rightarrow \text{stable}$$

3. $\dot{x} = y$
   $\dot{y} = -kx - \epsilon y^3(1 + x^2)$ ← Nonlinear damping, still non-hyperbolic

$$\dot{V} = kx(y) + y(-kx - \epsilon y^3(1 + x^2))$$
$$= -\epsilon y^4(1 + x^2)$$
$$\leq 0$$

(NOT $= 0$ $\forall (x,y) \in \mathbb{R}^2 \setminus (0,0)$ since $x = -1$ ... Then $\dot{V} = 0$ ...

$\Rightarrow$ stable (For $\epsilon > 0$)

Can show (soon) asy. stable using LaSalle's Invariance Principle

NOTE: ...en with $\dot{y} = -kx - \epsilon y$ ...en $\dot{V} = -\epsilon y^2$ so don't get asy. stable using energy as Lyap, even tho we know its true...

4. $\dot{x} = -x + 2y^3 - 2y^4$, $DF(0) = \begin{bmatrix} -1 & 0 \\ -1 & -1 \end{bmatrix}$
   $\dot{y} = -x - y + xy$

$\Rightarrow$ stable. But globally ??

Global stability

Thm (Barbashin - Krasovskii)

Suppose $F \in C^1(\mathbb{R}^n)$, $F(0)=0$, and $\exists$ real-valued $V \in C^1(\mathbb{R}^n)$ with $V(0)=0$, $V(x)>0$ if $x \neq x_0$, $\underbrace{V(x) \to \infty \text{ for } |x| \to \infty}_{\text{ensures level sets of } V \text{ are bounded}}$ and $\dot{V}(x) < 0 \ \forall x \neq 0$

Then $x=0$ is globally asymptotically stable.

Remark: Need extra condition so Ω_c is bounded for any c

$\forall c>0, \exists r>0$ s.t. $\|x\|>r \Rightarrow V(x)>c$, so $\Omega_c \subset B_r$

Example
$$\dot{x} = -x + 2y^3 - 2y^4$$
$$\dot{y} = -x - y + xy$$

$\Rightarrow$ Eq. pts if $x = 2y^4 - 2y^3$ & $2y^3 - 2y^4 - y + 2y^5 - 2y^4 = 0$

so 4 roots other than zero ?
(No... all are complex)

Try $V = x^{2m} + a y^{2n}$

So $\dot{V} = 2m x^{2m-1}(-x + 2y^3 - 2y^4) + 2an y^{2n-1}(-x - y + xy)$

Note $\overset{a=}{m} = n = 1$ gives
$$\dot{V} = -2x^2 + 4xy^3 - 4xy^4 - 2xy - 2y^2 + 2xy^2 \ \dots \ ??$$

w/ $m=1$, $n=2$:
$$\dot{V} = -2x^2 + 4xy^3 - 4xy^4 - 4ay^3x + 4ay^4x$$

Let $a=1 \Rightarrow \dot{V} = -2x^2 - 4y^4$

So $V = x^2 + y^4$ proves global asymp. stability.

(aside, no other eq. pts, and f is C^∞)

Boundedness:

If $\exists V$ satisfying $V(x) \to \infty$ for $|x| \to \infty$ (all level sets bdd) and $\dot{V}(x) \le 0 \ \forall x$, Then all trajectories are bounded

(ie $\forall x, \exists R \ni \|x(t)\| \le R \ \forall t \ge 0$)

(ie. every level set is invariant)

LaSalle's Invariance Principle

Let $\Omega \subset E$, compact set, positively invariant w.r.t. flow φ_t ← once in set, stay there. $x \in \Omega \Rightarrow \varphi_t(x) \in \Omega$, $t \geq 0$

Suppose $\exists$ real-valued $V \in C^1(E)$

with $\dot{V}(x) \leq 0$ in Ω.

Let $D_0 = \{x \in \Omega \mid \dot{V}(x) = 0\}$

and M the largest invariant set in D_0 $\left(M = \{x \in D_0 \mid \varphi_t(x) \in M \ \forall t\} \right)$

Then $\forall x \in \Omega$, $\lim_{t \to \infty} \varphi_t(x) \in M$

(every sol'n starting in Ω converges to M)

Remarks

- If $M = \{0\}$ then origin is asymp. stable (and Ω is a RoA)
  - → Can often use this to prove asy. stability using a Lyap. Fn with only $\dot{V}(x) \leq 0$
- Thm does not require $V \geq 0$

Example 3 from before:

$$\dot{x} = y$$
$$\dot{y} = -Kx - \epsilon y^3(1+x^2)$$
$$\dot{V} = -\epsilon y^4(1+x^2)$$
$$\leq 0$$

(or $\dot{y} = -Kx - \epsilon y$, $\dot{V} = -\epsilon y^2$

$\dot{V} = 0$ req. $y = 0$ $\Rightarrow \dot{x} = 0$. Only stay in D_0 if $\dot{y} = 0 \Rightarrow x = 0$)

But $\dot{V} = 0$ requires $y = 0$ or $x = \pm 1$

$D_0 = \{(x,y) : y = 0 \text{ or } x = \pm 1\}$

i) $y = 0$: Then $\dot{x} = 0$

Invariance requires $\dot{y} = 0 \Rightarrow x = 0$

ii) $x = \pm 1$, invariance requires $\dot{x} = 0 \Rightarrow y = 0 \ldots$

so $M = \{0, 0\}$ and origin is asy. stable.

(Ex 3b. If $\dot{y} = -\sin x - \epsilon y$

Then $\dot{y} = 0 \Rightarrow x = k\pi$

Need to restrict to $E = (-\pi, \pi) \times \mathbb{R}$ to get $M = \{(0,0)\}$ and that origin is asy. stable)

Equilibrium Points in $\mathbb{R}^2$

Hyperbolic: Node (2 real eig, same sign)

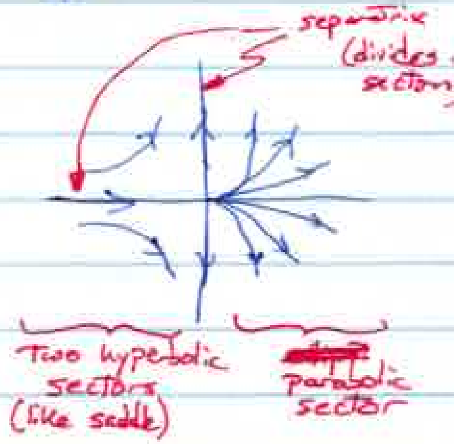

Saddle (" , opp. sign)


Focus (complex eig)

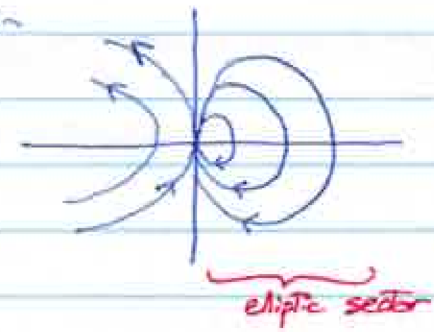

Non-hyperbolic: Center


Saddle-Node (One zero eig)

eg $\dot{y} = y$
$\dot{x} = x^2$



Critical point w/ elliptic domain



Cusp

Remarks

1. Non-hyperbolic, more variety!
2. Can get much more complex behaviour depending on leading non-zero order of diff eqn.
3. See non-hyperbolic behaviour at bifurcation points of dynamics (some "mode" transitions from stable to unstable. small perturb'n in parameters leads to different behaviour)
4. $\dot{x} = \alpha x^n$ stable if $\alpha < 0$, n odd

Centre Manifold Theory

Thm: Let $F \in C^r(E)$, $r \geq 1$, $E \subset \mathbb{R}^n$ containing origin,

$F(0) = 0$ and

$DF(0)$ with c eigenvalues with zero real part

$s = n - c$ " " $-$ve " "

Write the system as (2)

$$\dot{x} = Cx + F(x,y)$$
$$\dot{y} = Py + G(x,y)$$

$$\left(\dot{\xi} = F(\xi) = DF(0)\xi + \dots = \begin{bmatrix} C & 0 \\ 0 & P \end{bmatrix}\xi + \dots \right)$$

with

$(x,y) \in \mathbb{R}^c \times \mathbb{R}^s$, C square w/ c eig. zero real part

P " " s " $-$ve " "

and $F(0) = G(0) = 0$, $DF(0) = DG(0) = 0$

local only

Then $\exists \delta > 0$, function $h \in C^r(N_\delta(0))$, $h(0) = 0$, $Dh(0) = 0$

That defines the local centre manifold

Define manifold by constraint as before

$$W^c(0) = \{ (x,y) \in \mathbb{R}^c \times \mathbb{R}^s \mid y = h(x) \text{ for } |x| < \delta \}$$

and

That is the manifold is invariant wrt flow

$$Dh(x)\left[Cx + F(x, h(x))\right] = Ph(x) + G(x, h(x)) \qquad (3)$$

for $|x| < \delta$

And the flow on the centre manifold $W^c(0)$ is defined by

Only need to look at lower dimensional system!

$$\dot{x} = Cx + F(x, h(x)) \quad \forall x \in \mathbb{R}^c, \|x\| < \delta \qquad (4)$$

$\blacksquare$

More generally, if $f(0) = 0$ and $DF(0) = \begin{bmatrix} C & 0 & 0 \\ 0 & P & 0 \\ 0 & 0 & Q \end{bmatrix}$

where $C \in \mathbb{R}^{c \times c}$ has c eig w/ zero real part
$P \in \mathbb{R}^{s \times s}$ " s " " -ve " "
$Q \in \mathbb{R}^{u \times u}$ " u " " +ve " " , $u + s + c = n$

Then the flow in a nbhd of the origin of

$$\dot{x} = Cx + F(x, y, z)$$
$$\dot{y} = Py + G(x, y, z)$$
$$\dot{z} = Qz + H(x, y, z)$$

is topologically equiv. to the flow of

$$\dot{x} = Cx + F(x, h_1(x), h_2(x)) \qquad \text{for some } h_1(x),\ h_2(x)$$
$$\dot{y} = Py$$
$$\dot{z} = Qz$$

That is, H-G applies to parts w/ +ve or -ve eigenvalues, and to understand rest of the dynamics, only need to consider flow along c-dimensional centre manifold.

Approach

1. Convert to block diagonal form ②
2. Write $h(x)$ as polynomial series expansion
   $h(x) = ax^2 + bx^3 + \dots$
3. Solve for coefficients using ③, i.e. constraint that it is an invariant manifold. (Note, degree of accuracy limited by r)
4. Substitute $h(x)$ into ④ to determine flow along center (sic) manifold.

It might not...

5. If Taylor series for $h(x)$ converges then analytic cent man. $y = h(x)$ unique

Examples

1. $\dot{x} = x^2 y - x^5$ Note $\dot{y} = \ldots$ is stable, so expect $y \to 0$
$\dot{y} = -y + x^2$ If $y = 0$, $\dot{x} = -x^5$ (stable)

Here $C = 0$, $P = -1$, $F = x^2 y - x^5$, $G = x^2$

$h(x) = ax^2 + bx^3 + \ldots$, $Dh(x) = 2ax + 3bx^2 + \ldots$

$(2ax + 3bx^2)(ax^4 + bx^5 + \ldots - x^5) = -ax^2 - bx^3 - \ldots + x^2$

$\Rightarrow a = 1, b = 0, \ldots$ so $h(x) = x^2 + \ldots$

Substitute into $\dot{x} = x^2 y - x^5 = +x^4 - x^5 + \ldots$
$\Rightarrow$ Unstable !

Thm : For system (2), define for $\phi(x) \in C^1(E)$, $\phi(0) = 0$, $D\phi(0)$
$$(M\phi)(x) = D\phi(x)\left[Cx + F(x, \phi(x))\right] - P\phi(x) - G(x, \phi(x))$$
If for $x \to 0$
$$(M\phi)(x) = O(|x|^q) \quad \text{for some } q > 1$$
Then
$$|h(x) - \phi(x)| = O(|x|^q) \quad \text{as } x \to 0$$

Note re proof of C.M.T :

- Can find α distinct from real part of any eigenvalue
- Divide into eig w/ real part $> \alpha$ and real $< \alpha$
- Compute "pseudo-stable" and "pseudo-unstable manifolds
  (so $e^{-\alpha t} x(t) \to 0$ as $t \to \infty$ or $\to 0$ as $t \to -\infty$)
- Now choose a) $\min_{Re(\lambda)>0} Re(\lambda) > \alpha_1 > 0$ and b) $\max_{Re(\lambda)<0} < \alpha_2 < 0$

  and find a) U, W^{cs} and b) S, W^{cu}
- Then centre manifold is $W^{cs} \cap W^{cu}$

Example 2

$$\dot{x} = xy + ax^3 + by^2x$$
$$\dot{y} = -y + cx^2 + dx^2y$$

We know $\exists$ centre manifold tangent to x-axis at origin, locally of the form $y = h(x)$ and dynamics on c.m. given by $\dot{x} = xh(x) + ax^3 + b[h(x)]^2 x$

i) Start with $\phi(x) = \alpha x^2$ $\quad \left(h(x) = \alpha x^2 + O(x^3)\right)$
$[\phi(0) = 0,\ D\phi(0) = 0]$

Then $M\phi = 2\alpha x\left(\alpha x^3 + ax^3 + \alpha^2 b x^5\right) - \left(-\alpha x^2 + cx^2 + \alpha d x^4\right)$
$= -(c-\alpha)x^2 + O(x^3)$

So $h(x) = cx^2 + O(x^3)$

and $\dot{x} = (a+c)x^3 + O(x^4)$

$\Rightarrow$ stable if $a+c<0$
Unstable " $a+c>0$

Need higher order approx. if $a+c=0$