CDS140a
Nonlinear Systems: Local Theory
02/03/2011

4 Center Manifold Theory

Theorem (Local Center Manifold Theorem) Let $f \in C^r(E)$, where E is an open subset of $\mathbb{R}^n$ containing the origin and $r \geq 1$. Suppose that $f(0) = 0$ and that $Df(0)$ has c eigenvalues with zero real part, and $s = n - c$ eigenvalues with negative real part. The system (1) then can be written in diagonal form

$$\begin{aligned} \dot{x} &= Cx + F(x,y) \\ \dot{y} &= Py + G(x,y) \end{aligned} \tag{4}$$

where $(x,y) \in \mathbb{R}^c \times \mathbb{R}^s$, C is a square matrix with c eigenvalues with zero real parts, P is a square matrix with s eigenvalues with negative real parts, and $F(0) = G(0) = 0$, $DF(0) = DG(0) = 0$; furthermore, there exists a $\delta > 0$ and a function $h \in C^r(N_\delta(0))$, $h(0) = 0$, $Dh(0) = 0$ the defines the local center manifold $W^c(0) := \{(x,y) \in \mathbb{R}^c \times \mathbb{R}^s \mid y = h(x) \text{ for } |x| < \delta\}$ and satisfies

$$Dh(x)\left[Cx + F(x,h(x))\right] = Ph(x) + G(x,h(x)) \tag{5}$$

for $|x| < \delta$; and the flow on the center manifold $W^c(0)$ is defined by the system of differential equations

$$\dot{x} = Cx + F(x,h(x)) \tag{6}$$

for all $x \in \mathbb{R}^c$ with $|x| < \delta$.

This theorem can be used to determine the flow near nonhyperbolic equilibrium points. The strategy is:

1. Convert (1) in diagonal form (4)
2. Use a series expansion for the components of $h(x)$ (up to the degree of accuracy we need, provided that r is sufficiently large)
3. Determine the components of the expansion of $h(x)$ using (5)
4. Substitute this approximate expression of $h(x)$ into (6) to determine the flow.

Example: Perko 2.12.3 Use the above theorem to determine the qualitative behavior of the origin for the system

$$\begin{aligned} \dot{x} &= xy \\ \dot{y} &= -y - x^2 \end{aligned}$$

Solution: The system is already in the desired form, with $C = 0$, $P = -1$, $F(x,y) = xy$ and $G(x,y) = -x^2$. Let

$$\begin{aligned} h(x) &= ax^2 + bx^3 + \cdots \\ Dh(x) &= 2ax + 3bx^2 + \cdots \\ Dh(x)\left[Cx + F(x,h(x))\right] &= \left(2ax + 3bx^2 + \cdots\right) x \left(ax^2 + bx^3 + \cdots\right) \\ Ph(x) + G(x,h(x)) &= -\left(ax^2 + bx^3 + \cdots\right) - x^2 \\ &\Downarrow \text{ Collecting terms (Using (5))} \\ O(x^2) &: \; -a - 1 = 0 \\ O(x^3) &: \; b = 0 \\ &\vdots \end{aligned}$$

So $h(x) = -x^2 + O(x^4)$. The flow on the center manifold is given by (6)

$$\dot{x} = F(x, h(x)) = -x^3 + O(x^5)$$

i.e., the system is stable.

Example: Perko 2.12.5(a) Use the above theorem to determine the qualitative behavior of the origin for the system

$$\begin{aligned} \dot{x}_1 &= -x_2 + x_1 y \\ \dot{x}_2 &= x_1 + x_2 y \\ \dot{y} &= -y - x_1^2 - x_2^2 + y^2 \end{aligned}$$

Solution: The system is already in the desired form, with $C = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$, $P = -1$, $F(x, y) = \begin{bmatrix} x_1 y \\ x_2 y \end{bmatrix}$ and $G(x, y) = -x_1^2 - x_2^2 + y^2$. Let

$$\begin{aligned} h(x_1, x_2) &= ax_1^2 + bx_1x_2 + cx_2^2 + \cdots \\ Dh(x) &= [2ax_1 + bx_2 + \cdots, \quad bx_1 + 2cx_2 \cdots] \\ Dh(x)\,[Cx + F(x, h(x))] &= \begin{bmatrix} 2ax_1 + bx_2 + \cdots, & bx_1 + 2cx_2 \cdots \end{bmatrix} \begin{bmatrix} -x_2 + x_1\left(ax_1^2 + bx_1x_2 + cx_2^2 + \cdots\right) \\ x_1 + x_2\left(ax_1^2 + bx_1x_2 + cx_2^2 + \cdots\right) \end{bmatrix} \\ Ph(x) + G(x, h(x)) &= -\left(ax_1^2 + bx_1x_2 + cx_2^2 + \cdots\right) - x_1^2 - x_2^2 + \left(ax_1^2 + bx_1x_2 + cx_2^2 + \cdots\right)^2 \\ &\Downarrow \text{ Collecting terms (Using (5))} \\ x_1^2 &: \ b = -a - 1 \\ x_2^2 &: \ -b = -c - 1 \\ x_1x_2 &: \ -2a + 2c = -b \\ &\vdots \end{aligned}$$

we get $a = -1$, $b = 0$, $c = -1$, and so $h(x_1, x_2) = -x_1^2 - x_2^2 + O(|x|^3)$. The flow on the center manifold is given by (6)

$$\begin{aligned} \dot{x}_1 &= -x_2 + x_1\left(-x_1^2 - x_2^2 + O(|x|^3)\right) = -x_2 - x_1^3 - x_1x_2^2 + O(|x|^4) \\ \dot{x}_2 &= x_1 + x_2\left(-x_1^2 - x_2^2 + O(|x|^3)\right) = x_1 - x_2^3 - x_1^2x_2 + O(|x|^4) \end{aligned}$$

Note: The center manifold theorem allows us to determine the local behavior of the system by looking at the flow on a lower dimensional manifold, i.e., instead of working with an n-dimensional system, we can just deal with a c-dimensional one (i.e., we "reduce" the full n-dimensional system into a c-dimensional system). In this example, instead of trying to determine the behavior of the full 3-d system, we reduced the problem to determine the behavior of the 2-d system above.

To determine the stability of the origin, let's try changing to polar coordinates

$$\begin{aligned} r\dot{r} &= x_1\dot{x}_1 + x_2\dot{x}_2 = x_1\left(-x_2 - x_1^3 - x_1x_2^2\right) + x_2\left(x_1 - x_2^3 - x_1^2x_2\right) + O(|r|^5) \\ &= -x_1^4 - 2x_1^2x_2^2 - x_2^4 + O(|r|^5) = -\left(x_1^2 + x_2^2\right)^2 + O(|r|^5) = -r^4 + O(|r|^5) \\ \dot{r} &= -r^3 + O(|r|^4) \end{aligned}$$

i.e., the system is stable.

Example: Perko 2.12.6 Use the above theorem to determine the qualitative behavior of the origin for the system

$$\begin{aligned} \dot{x} &= ax^2 + bxy + cy^2 \\ \dot{y} &= -y + dx^2 + exy + fy^2 \end{aligned}$$

Solution: The system is already in the desired form, with $C = 0$, $P = -1$, $F(x, y) = ax^2 + bxy + cy^2$ and $G(x, y) = dx^2 + exy + fy^2$. Let

$$\begin{aligned}
h(x) &= k_2x^2 + k_3x^3 + \cdots \\
Dh(x) &= 2k_2x + 3k_3x^2 + \cdots \\
Dh(x)\left[Cx + F(x, h(x))\right] &= \left(2k_2x + 3k_3x^2 + \cdots\right)\left(ax^2 + bx\left(k_2x^2 + k_3x^3 + \cdots\right) + c\left(k_2x^2 + k_3x^3 + \cdots\right)^2\right) \\
Ph(x) + G(x, h(x)) &= -\left(k_2x^2 + k_3x^3 + \cdots\right) + dx^2 + ex\left(k_2x^2 + k_3x^3 + \cdots\right) + f\left(k_2x^2 + k_3x^3 + \cdots\right)^2 \\
&\Downarrow \text{ Collecting terms (Using (5))} \\
O(x^2) &: -k_2 + d = 0 \\
O(x^3) &: 2k_2a = -k_3 + ek_2 \Rightarrow k_3 = (e - 2a)k_2 \\
&\vdots
\end{aligned}$$

So $h(x) = dx^2 + (e - 2a)dx^3 + O(x^4)$. The flow on the center manifold is given by (6)

$$\begin{aligned}
\dot{x} = F(x, h(x)) &= ax^2 + bx\left(dx^2 + (e - 2a)dx^3 + O(x^4)\right) + c\left(dx^2 + (e - 2a)dx^3 + O(x^4)\right)^2 \\
&= ax^2 + bdx^3 + \left(bd(e - 2a) + cd^2\right)x^4 + O(x^5)
\end{aligned}$$

So unstable for $a \neq 0$. For $a = 0$ and $bd > 0$ unstable, $bd < 0$ stable. For $a = 0$, $bd = 0$, $c \neq 0$ it is unstable.

Example: Perko 2.12.4 Use the above theorem to determine the qualitative behavior of the origin for the system

$$\begin{aligned}
\dot{x} &= -x^3 \\
\dot{y} &= -y + x^2
\end{aligned}$$

Solution: First, it is clear that the system is stable since the flow on the center manifold is $\dot{x} = -x^3$. Let's calculate the center manifold anyway. The system is already in the desired form, with $C = 0$, $P = -1$, $F(x, y) = -x^3$ and $G(x, y) = x^2$. Let

$$\begin{aligned}
h(x) &= a_2x^2 + a_3x^3 + \cdots \\
Dh(x) &= 2a_2x + 3a_3x^2 + \cdots \\
Dh(x)\left[Cx + F(x, h(x))\right] &= -\left(2a_2x + 3a_3x^2 + \cdots\right)x^3 \\
Ph(x) + G(x, h(x)) &= -\left(a_2x^2 + a_3x^3 + \cdots\right) + x^2 \\
&\Downarrow \text{ Collecting terms (Using (5))} \\
O(x^2) &: -a_2 + 1 = 0 \\
O(x^3) &: a_3 = 0 \\
O(x^4) &: -2a_2 = -a_4 \Rightarrow a_4 = 2
\end{aligned}$$

In general, $a_{2k+1} = 0$ and $a_{n+2} = na_n$for n even. So the Taylor series $x^2 + 2x^4 + 8x^6 + \cdots$ which diverges for $x \neq 0$. What happened? Let's try to solve for $h(x)$ without using the power expansion

$$\begin{aligned}
\frac{dh(x)}{dx}(-x^3) &= -h(x) + x^2 \\
&\Downarrow \\
h' &= \frac{1}{x^3}h - \frac{1}{x}
\end{aligned}$$

The existence and uniqueness theorem does not apply (since the vector field is not continous in x at 0) . Indeed using mathematica to solve, it appears to give a continuoum of solutions (all satisfying $h(0) = 0$,

$h'(0) = 0$),

$$h(x) = \exp\left(-\frac{1}{2x^2}\right) a + \frac{1}{2}\exp\left(-\frac{1}{2x^2}\right) ExpIntegralEi\left(\frac{1}{2x^2}\right)$$

, where a is some constant and

$$ExpIntegralEi(z) = Ei(z) = -\int_{-z}^{\infty} \frac{e^{-t}}{t} dt.$$

<u>*Remarks:*</u>

- There may be many functions $h(x)$ that determine diferent center manifolds. However, the flows on the different center manifolds are determined by 6 and are all topologically equivalent near the origin.
- For an analytical system, if the series expansion of h converges, then there exists a unique analytical center manifold.
- For a analytical system (even polynomial), if the series does not converge, then an analytical center manifold need not exist.

Theorem Let $f \in C^1(E)$, where E is an open subset of $\mathbb{R}^n$ containing the origin. Suppose that $f(0) = 0$ and that $Df(0) = diag\,[C, P, Q]$, where the square matric C has c eigenvalues with zero real part, the square matric P has s eigenvalues with negative real part, and the square matric Q has $u = n - c - s$ eigenvalues with positive real part. Then there exist C^1 functions $h_1(x)$, $h_2(x)$ satisfying

$$\begin{aligned} Dh_1(x)\,[Cx + F(x, h_1(x), h_2(x))] &= Ph_1(x) + G(x, h_1(x), h_2(x)) \\ Dh_2(x)\,[Cx + F(x, h_1(x), h_2(x))] &= Qh_2(x) + H(x, h_1(x), h_2(x)) \end{aligned}$$

in a neighborhood of the origin such that the nonlinear system (1) which can be writen as

$$\begin{aligned} \dot{x} &= Cx + F(x, y, z) \\ \dot{y} &= Py + G(x, y, z) \\ \dot{z} &= Qz + H(x, y, z) \end{aligned}$$

is topologically conjugate to the C^1system

$$\begin{aligned} \dot{x} &= Cx + F(x, h_1(x), h_2(x)) \\ \dot{y} &= Py \\ \dot{z} &= Qz \end{aligned}$$

for $(x, y, z) \in \mathbb{R}^c \times \mathbb{R}^s \times \mathbb{R}^u$ in a neighbrhood of the origin.

References

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[3] S. H. Strogatz. *Nonlinear Dynamics And Chaos: With Applications To Physics, Biology, Chemistry, And Engineering.* Westview Press, 2001.