Chapter 6

Design Constraints

Before we see how to design control systems for the robust performance specification, it is useful to determine the basic limitations on achievable performance. In this chapter we study design constraints arising from two sources: from algebraic relationships that must hold among various transfer functions; from the fact that closed-loop transfer functions must be stable (i.e., analytic in the right half-plane). It is assumed throughout this chapter that the feedback system is internally stable.

6.1 Algebraic Constraints

There are three items in this section.

1. The identity $S + T = 1$ always holds. This is an immediate consequence of the definitions of S and T. So in particular, $|S(j\omega)|$ and $|T(j\omega)|$ cannot both be less than 1/2 at the same frequency ω.

2. A necessary condition for robust performance is that the weighting functions satisfy

$$\min\{|W_1(j\omega)|, |W_2(j\omega)|\} < 1, \qquad \forall \omega.$$

Proof Fix ω and assume that $|W_1| \le |W_2|$ (the argument $j\omega$ is suppressed). Then

$$\begin{aligned} |W_1| &= |W_1(S+T)| \\ &\le |W_1 S| + |W_1 T| \\ &\le |W_1 S| + |W_2 T|. \end{aligned}$$

So robust performance (see Theorem 4.2), that is,

$$\| |W_1 S| + |W_2 T| \|_\infty < 1,$$

implies that $|W_1| < 1$, and hence

$$\min\{|W_1|, |W_2|\} < 1.$$

The same conclusion can be drawn when $|W_2| \le |W_1|$. ■

So at every frequency either $|W_1|$ or $|W_2|$ must be less than 1. Typically, $|W_1(j\omega)|$ is monotonically decreasing—for good tracking of low-frequency signals—and $|W_2(j\omega)|$ is monotonically increasing—uncertainty increases with increasing frequency.

3. If p is a pole of L in $\mathrm{Re}s \geq 0$ and z is a zero of L in the same half-plane, then

$$S(p) = 0, \quad S(z) = 1, \tag{6.1}$$

$$T(p) = 1, \quad T(z) = 0. \tag{6.2}$$

These interpolation constraints are immediate from the definitions of S and T. For example,

$$S(p) = \frac{1}{1 + L(p)} = \frac{1}{\infty} = 0.$$

6.2 Analytic Constraints

In this section we derive some constraints concerning achievable performance obtained from analytic function theory. The first subsection presents some mathematical preliminaries.

Preliminaries

We begin with the following fundamental facts concerning complex functions: the maximum modulus theorem, Cauchy's theorem, and Cauchy's integral formula. These are stated here for convenience.

Maximum Modulus Theorem *Suppose that Ω is a region (nonempty, open, connected set) in the complex plane and F is a function that is analytic in Ω. Suppose that F is not equal to a constant. Then $|F|$ does not attain its maximum value at an interior point of Ω.*

A simple application of this theorem, with Ω equal to the open right half-plane, shows that for F in $\mathcal{S}$

$$\|F\|_\infty = \sup_{\mathrm{Re}s>0} |F(s)|.$$

Cauchy's Theorem *Suppose that Ω is a bounded open set with connected complement and $\mathcal{D}$ is a non-self-intersecting closed contour in Ω. If F is analytic in Ω, then*

$$\oint_{\mathcal{D}} F(s)ds = 0.$$

Cauchy's Integral Formula *Suppose that F is analytic on a non-self-intersecting closed contour $\mathcal{D}$ and in its interior Ω. Let s_0 be a point in Ω. Then*

$$F(s_0) = \frac{1}{2\pi j} \oint_{\mathcal{D}} \frac{F(s)}{s - s_0} ds.$$

We shall also need the Poisson integral formula, which says that the value of a bounded analytic function at a point in the right half-plane is completely determined by the coordinates of the point together with the values of the function on the imaginary axis.

Lemma 1 *Let F be analytic and of bounded magnitude in $\mathrm{Re}s \geq 0$ and let $s_0 = \sigma_0 + j\omega_0$ be a point in the complex plane with $\sigma_0 > 0$. Then*

$$F(s_0) = \frac{1}{\pi} \int_{-\infty}^{\infty} F(j\omega) \frac{\sigma_0}{\sigma_0^2 + (\omega - \omega_0)^2} d\omega.$$

Proof Construct the Nyquist contour $\mathcal{D}$ in the complex plane taking the radius, r, large enough so that the point s_0 is encircled by $\mathcal{D}$.

Cauchy's integral formula gives

$$F(s_0) = \frac{1}{2\pi j}\oint_{\mathcal{D}} \frac{F(s)}{s - s_0} ds.$$

Also, since $-\overline{s}_0$ is not encircled by $\mathcal{D}$, Cauchy's theorem gives

$$0 = \frac{1}{2\pi j}\oint_{\mathcal{D}} \frac{F(s)}{s + \overline{s}_0} ds.$$

Subtract these two equations to get

$$F(s_0) = \frac{1}{2\pi j}\oint_{\mathcal{D}} F(s)\frac{\overline{s}_0 + s_0}{(s - s_0)(s + \overline{s}_0)} ds.$$

Thus

$$F(s_0) = I_1 + I_2,$$

where

$$\begin{aligned} I_1 &:= \frac{1}{\pi}\int_{-r}^{r} F(j\omega)\frac{\sigma_0}{(s_0 - j\omega)(\overline{s}_0 + j\omega)} d\omega \\ &= \frac{1}{\pi}\int_{-r}^{r} F(j\omega)\frac{\sigma_0}{\sigma_0^2 + (\omega - \omega_0)^2} d\omega, \\ I_2 &:= \frac{1}{\pi j}\int_{-\pi/2}^{\pi/2} F(re^{j\theta})\frac{\sigma_0}{(re^{j\theta} - s_0)(re^{j\theta} + \overline{s}_0)} rje^{j\theta} d\theta. \end{aligned}$$

As $r \to \infty$

$$I_1 \to \frac{1}{\pi}\int_{-\infty}^{\infty} F(j\omega)\frac{\sigma_0}{\sigma_0^2 + (\omega - \omega_0)^2} d\omega.$$

So it remains to show that $I_2 \to 0$ as $r \to \infty$.

We have

$$I_2 \le \frac{\sigma_0}{\pi}\|F\|_\infty \frac{1}{r}\int_{-\pi/2}^{\pi/2} \frac{1}{|e^{j\theta} - s_0 r^{-1}||e^{j\theta} + \overline{s}_0 r^{-1}|} d\theta.$$

The integral

$$\int_{-\pi/2}^{\pi/2} \frac{1}{|e^{j\theta} - s_0 r^{-1}||e^{j\theta} + \overline{s}_0 r^{-1}|} d\theta$$

converges as $r \to \infty$. Thus

$$I_2 \le \text{constant } \times \frac{1}{r},$$

which gives the desired result. ■

Bounds on the Weights W_1 and W_2

Suppose that the loop transfer function L has a zero z in $\mathrm{Re}s \geq 0$. Then

$$\|W_1 S\|_\infty \geq |W_1(z)|. \tag{6.3}$$

This is a direct consequence of the maximum modulus theorem and (6.1):

$$|W_1(z)| = |W_1(z)S(z)| \leq \sup_{\mathrm{Re}s \geq 0} |W_1(s)S(s)| = \|W_1 S\|_\infty.$$

So a necessary condition that the performance criterion $\|W_1 S\|_\infty < 1$ be achievable is that the weight satisfy $|W_1(z)| < 1$. In words, the magnitude of the weight at a right half-plane zero of P or C must be less than 1.

Similarly, suppose that L has a pole p in $\mathrm{Re}s \geq 0$. Then

$$\|W_2 T\|_\infty \geq |W_2(p)|, \tag{6.4}$$

so a necessary condition for the robust stability criterion $\|W_2 T\|_\infty < 1$ is that the weight W_2 satisfy $|W_2(p)| < 1$.

All-Pass and Minimum-Phase Transfer Functions

Two types of transfer functions play a critical role in the rest of this book: all-pass and minimum-phase. A function in $\mathcal{S}$ is *all-pass* if its magnitude equals 1 at all points on the imaginary axis. The terminology comes from the fact that a filter with an all-pass transfer function passes without attenuation input sinusoids of all frequencies. It is not difficult to show that such a function has pole-zero symmetry about the imaginary axis in the sense that a point s_0 is a zero iff its reflection, $-\overline{s}_0$, is a pole. Consequently, the function being stable, all its zeros lie in the right half-plane. Thus an all-pass function is, up to sign, the product of factors of the form

$$\frac{s - s_0}{s + \overline{s}_0}, \qquad \mathrm{Re}s_0 > 0.$$

Examples of all-pass functions are

$$1, \quad \frac{s-1}{s+1}, \quad \frac{s^2 - s + 2}{s^2 + s + 2}.$$

A function in $\mathcal{S}$ is *minimum-phase* if it has no zeros in $\mathrm{Re}s > 0$. This terminology can be explained as follows. Let G be a minimum-phase transfer function. There are many other transfer functions having the same magnitude as G, for example FG where F is all-pass. But all these other transfer functions have greater phase. Thus, of all the transfer functions having Gs magnitude, the one with the minimum phase is G. Examples of minimum-phase functions are

$$1, \quad \frac{1}{s+1}, \quad \frac{s}{s+1}, \quad \frac{s+2}{s^2+s+1}.$$

It is a useful fact that every function in $\mathcal{S}$ can be written as the product of two such factors: for example

$$\frac{4(s-2)}{s^2+s+1} = \left(\frac{s-2}{s+2}\right)\left(\frac{4(s+2)}{s^2+s+1}\right).$$

Lemma 2 *For each function G in $\mathcal{S}$ there exist an all-pass function G_{ap} and a minimum-phase function G_{mp} such that $G = G_{ap}G_{mp}$. The factors are unique up to sign.*

Proof Let G_{ap} be the product of all factors of the form

$$\frac{s - s_0}{s + \overline{s}_0},$$

where s_0 ranges over all zeros of G in $\mathrm{Re}\, s > 0$, counting multiplicities, and then define

$$G_{mp} = \frac{G}{G_{ap}}.$$

The proof of uniqueness is left as an exercise. ■

For technical reasons we assume for the remainder of this section that L has no poles on the imaginary axis. Factor the sensitivity function as

$$S = S_{ap}S_{mp}.$$

Then S_{mp} has no zeros on the imaginary axis (such zeros would be poles of L) and S_{mp} is not strictly proper (since S is not). Thus $S_{mp}^{-1} \in \mathcal{S}$.

As a simple example of the use of all-pass functions, suppose that P has a zero at z with $z > 0$, a pole at p with $p > 0$; also, suppose that C has neither poles nor zeros in the closed right half-plane. Then

$$S_{ap}(s) = \frac{s - p}{s + p}, \quad T_{ap}(s) = \frac{s - z}{s + z}.$$

It follows from the preceding section that $S(z) = 1$, and hence

$$S_{mp}(z) = S_{ap}(z)^{-1} = \frac{z + p}{z - p}.$$

Similarly,

$$T_{mp}(p) = T_{ap}(p)^{-1} = \frac{p + z}{p - z}.$$

Then

$$\|W_1 S\|_\infty = \|W_1 S_{mp}\|_\infty \geq |W_1(z) S_{mp}(z)| = \left| W_1(z) \frac{z + p}{z - p} \right|$$

and

$$\|W_2 T\|_\infty \geq \left| W_2(p) \frac{p + z}{p - z} \right|.$$

Thus, if there are a pole and zero close to each other in the right half-plane, they can greatly amplify the effect that either would have alone.

Example These inequalities are effectively illustrated with the cart-pendulum example of Section 5.7. Let $P(s)$ be the u-to-x transfer function for the up position of the pendulum, that is,

$$P(s) = \frac{ls^2 - g}{s^2[Mls^2 - (M + m)g]}.$$

Define the ratio $r := m/M$ of pendulum mass to cart mass. The zero and pole of P in $\mathrm{Re}s > 0$ are

$$z = \sqrt{\frac{g}{l}}, \quad p = z\sqrt{1+r}.$$

Note that for r fixed, a larger value of l means a smaller value of p, and this in turn means that the system is easier to stabilize (the time constant is slower). The foregoing two inequalities on $\|W_1 S\|_\infty$ and $\|W_2 T\|_\infty$ apply. Since the cart-pendulum is a stabilization task, let us focus on

$$\|W_2 T\|_\infty \geq \left| W_2(p) \frac{p+z}{p-z} \right|. \tag{6.5}$$

The robust stabilization problem becomes harder the larger the value of the right-hand side of (6.5). The scaling factor in this inequality is

$$\frac{p+z}{p-z} = \frac{\sqrt{1+r}+1}{\sqrt{1+r}-1}. \tag{6.6}$$

This quantity is always greater than 1, and it approaches 1 only when r approaches ∞, that is, only when the pendulum mass is much larger than the cart mass. There is a tradeoff, however, in that a large value of r means a large value of p, the unstable pole; for a typical W_2 (high-pass) this in turn means a relatively large value of $|W_2(p)|$ in (6.5). So at least for small uncertainty, the worst-case scenario is a short pendulum with a small mass m relative to the cart mass M.

In contrast, the u-to-y transfer function has no zeros, so the constraint there is simply

$$\|W_2 T\|_\infty \geq |W_2(p)|.$$

If robust stabilization were the only objective, we could achieve equality by careful selection of the controller. Note that for this case there is no apparent tradeoff in making m/M large. The difference between the two cases, measuring x and measuring y, again highlights the important fact that sensor location can have a significant effect on the difficulty of controlling a system or on the ultimate achievable performance.

Some simple experiments can be done to illustrate the points made in this example. Obtain several sticks of various lengths and try to balance them in the palm of your hand. You will notice that it is easier to balance longer sticks, because the dynamics are slower and p above is smaller. It is also easier to balance the sticks if you look at the top of the stick (measuring y) rather than at the bottom (measuring x). In fact, even for a stick that is easily balanced when looking at the top, it will be impossible to balance it while looking only at the bottom. There is also feedback from the forces that your hand feels, but this is similar to measuring x.

The interested reader may repeat the analysis for the down position of the pendulum. At this point it is useful to include the following lemma which will be used subsequently.

Lemma 3 *For every point $s_0 = \sigma_0 + j\omega_0$ with $\sigma_0 > 0$,*

$$\log |S_{mp}(s_0)| = \frac{1}{\pi} \int_{-\infty}^{\infty} \log |S(j\omega)| \frac{\sigma_0}{\sigma_0^2 + (\omega - \omega_0)^2} d\omega.$$

Proof Set $F(s) := \ln S_{mp}(s)$. Then F is analytic and of bounded magnitude in $\mathrm{Re}s \geq 0$. (This follows from the properties $S_{mp}, S_{mp}^{-1} \in \mathcal{S}$; the idea is that since S_{mp} has no poles or zeros in the right half-plane, $\ln S_{mp}$ is well-behaved there.) Apply Lemma 1 to get

$$F(s_0) = \frac{1}{\pi} \int_{-\infty}^{\infty} F(j\omega) \frac{\sigma_0}{\sigma_0^2 + (\omega - \omega_0)^2} d\omega.$$

Now take real parts of both sides:

$$\mathrm{Re}F(s_0) = \frac{1}{\pi} \int_{-\infty}^{\infty} \mathrm{Re}F(j\omega) \frac{\sigma_0}{\sigma_0^2 + (\omega - \omega_0)^2} d\omega. \tag{6.7}$$

But

$$S_{mp} = \mathrm{e}^F = \mathrm{e}^{\mathrm{Re}F} \mathrm{e}^{j\mathrm{Im}F},$$

so

$$|S_{mp}| = \mathrm{e}^{\mathrm{Re}F},$$

that is,

$$\ln |S_{mp}| = \mathrm{Re}F.$$

Thus from (6.7)

$$\ln |S_{mp}(s_0)| = \frac{1}{\pi} \int_{-\infty}^{\infty} \ln |S_{mp}(j\omega)| \frac{\sigma_0}{\sigma_0^2 + (\omega - \omega_0)^2} d\omega,$$

or since $|S| = |S_{mp}|$ on the imaginary axis,

$$\ln |S_{mp}(s_0)| = \frac{1}{\pi} \int_{-\infty}^{\infty} \ln |S(j\omega)| \frac{\sigma_0}{\sigma_0^2 + (\omega - \omega_0)^2} d\omega.$$

Finally, since $\log x = \log \mathrm{e} \ln x$, the result follows upon multiplying the last equation by $\log \mathrm{e}$. ■

The Waterbed Effect

Consider a tracking problem where the reference signals have their energy spectra concentrated in a known frequency range, say $[\omega_1, \omega_2]$. This is the idealized situation where W_1 is a bandpass filter. Let M_1 denote the maximum magnitude of S on this frequency band,

$$M_1 := \max_{\omega_1 \leq \omega \leq \omega_2} |S(j\omega)|,$$

and let M_2 denote the maximum magnitude over all frequencies, that is, $\|S\|_\infty$. Then good tracking capability is characterized by the inequality $M_1 \ll 1$. On the other hand, we cannot permit M_2 to be too large: Remember (Section 4.2) that $1/M_2$ equals the distance from the critical point to the Nyquist plot of L, so large M_2 means small stability margin (a typical upper bound for M_2 is 2). Notice that M_2 must be at least 1 because this is the value of S at infinite frequency. So the question arises: Can we have M_1 very small and M_2 not too large? Or does it happen that very small M_1 necessarily means very large M_2? The latter situation might be compared to a waterbed: As $|S|$ is pushed down on one frequency range, it pops up somewhere else. It turns out that non-minimum-phase plants exhibit the waterbed effect.

Theorem 1 *Suppose that P has a zero at z with $Rez > 0$. Then there exist positive constants c_1 and c_2, depending only on ω_1, ω_2, and z, such that*

$$c_1 \log M_1 + c_2 \log M_2 \geq \log |S_{ap}(z)^{-1}| \geq 0.$$

Proof Since z is a zero of P, it follows from the preceding section that $S(z) = 1$, and hence $S_{mp}(z) = S_{ap}(z)^{-1}$. Apply Lemma 3 with

$$s_0 = z = \sigma_0 + j\omega_0$$

to get

$$\log |S_{ap}(z)^{-1}| = \frac{1}{\pi} \int_{-\infty}^{\infty} \log |S(j\omega)| \frac{\sigma_0}{\sigma_0^2 + (\omega - \omega_0)^2} d\omega.$$

Thus

$$\log |S_{ap}(z)^{-1}| \leq c_1 \log M_1 + c_2 \log M_2,$$

where c_1 is defined to be the integral of

$$\frac{1}{\pi} \frac{\sigma_0}{\sigma_0^2 + (\omega - \omega_0)^2}$$

over the set

$$[-\omega_2, -\omega_1] \cup [\omega_1, \omega_2]$$

and c_2 equals the same integral but over the complementary set.

It remains to observe that $|S_{ap}(z)| \leq 1$ by the maximum modulus theorem, so

$$\log |S_{ap}(z)^{-1}| \geq 0. \blacksquare$$

Example As an illustration of the theorem consider the plant transfer function

$$P(s) = \frac{s-1}{(s+1)(s-p)},$$

where $p > 0$, $p \neq 1$. As observed in the preceding section, S must interpolate zero at the unstable poles of P, so $S(p) = 0$. Thus the all-pass factor of S must contain the factor

$$\frac{s-p}{s+p}.$$

that is,

$$S_{ap}(s) = \frac{s-p}{s+p} G(s)$$

for some all-pass function G. Since $|G(1)| \leq 1$ (maximum modulus theorem), there follows

$$|S_{ap}(1)| \leq \left| \frac{1-p}{1+p} \right|.$$

So the theorem gives

$$c_1 \log M_1 + c_2 \log M_2 \geq \log \left| \frac{1+p}{1-p} \right|.$$

Note that the right-hand side is very large if p is close to 1. This example illustrates again a general fact: The waterbed effect is amplified if the plant has a pole and a zero close together in the right half-plane. We would expect such a plant to be very difficult to control.

It is emphasized that the waterbed effect applies to non-minimum-phase plants only. In fact, the following can be proved (Section 10.1): If P has no zeros in $\text{Re}s > 0$ nor on the imaginary axis in the frequency range $[\omega_1, \omega_2]$, then for every $\epsilon > 0$ and $\delta > 1$ there exists a controller C so that the feedback system is internally stable, $M_1 < \epsilon$, and $M_2 < \delta$. As a very easy example, take

$$P(s) = \frac{1}{s+1}.$$

The controller $C(s) = k$ is internally stabilizing for all $k > 0$, and then

$$S(s) = \frac{s+1}{s+1+k}.$$

So $\|S\|_\infty = 1$ and, for every $\epsilon > 0$ and ω_2, if k is large enough, then

$$|S(j\omega)| < \epsilon, \quad \forall \omega \le \omega_2.$$

The Area Formula

Herein is derived a formula for the area bounded by the graph of $|S(j\omega)|$ (log scale) plotted as a function of ω (linear scale). The formula is valid when the relative degree of L is large enough. *Relative degree* equals degree of denominator minus degree of numerator.

Let $\{p_i\}$ denote the set of poles of L in $\text{Re}s > 0$.

Theorem 2 *Assume that the relative degree of L is at least 2. Then*

$$\int_0^\infty \log|S(j\omega)|d\omega = \pi(\log \mathrm{e})(\sum \text{Re}p_i).$$

Proof In Lemma 3 take $\omega_0 = 0$ to get

$$\log|S_{mp}(\sigma_0)| = \frac{1}{\pi}\int_{-\infty}^\infty \log|S(j\omega)|\frac{\sigma_0}{\sigma_0^2+\omega^2}d\omega,$$

or equivalently,

$$\int_0^\infty \log|S(j\omega)|\frac{\sigma_0}{\sigma_0^2+\omega^2}d\omega = \frac{\pi}{2}\log|S_{mp}(\sigma_0)|.$$

Multiply by σ_0:

$$\int_0^\infty \log|S(j\omega)|\frac{\sigma_0^2}{\sigma_0^2+\omega^2}d\omega = \frac{\pi}{2}\sigma_0\log|S_{mp}(\sigma_0)|.$$

It can be shown that the left-hand side converges to

$$\int_0^\infty \log|S(j\omega)|d\omega$$

as $\sigma_0 \to \infty$. [The idea is that for very large σ_0 the function

$$\frac{\sigma_0^2}{\sigma_0^2+\omega^2}$$

equals nearly 1 up to large values of ω. On the other hand, $\log|S(j\omega)|$ tends to zero as ω tends to ∞.] So it remains to show that

$$\lim_{\sigma\to\infty}\frac{\sigma}{2}\log|S_{mp}(\sigma)| = (\log \mathrm{e})(\sum \text{Re}p_i). \tag{6.8}$$

We can write

$$S = S_{ap} S_{mp},$$

where

$$S_{ap}(s) = \prod_i \frac{s - p_i}{s + \overline{p}_i}.$$

It is claimed that

$$\lim_{\sigma \to \infty} \sigma \ln S(\sigma) = 0.$$

To see this, note that since L has relative degree at least 2 we can write

$$L(\sigma) \approx \frac{c}{\sigma^k} \text{ as } \sigma \to \infty$$

for some constant c and some integer $k \geq 2$. Thus as $\sigma \to \infty$

$$\sigma \ln S(\sigma) = -\sigma \ln[1 + L(\sigma)] \approx -\sigma \ln \left(1 + \frac{c}{\sigma^k}\right).$$

Now use the Maclaurin's series

$$\ln(1 + x) = x - \frac{x^2}{2} + \cdots \tag{6.9}$$

to get

$$\sigma \ln S(\sigma) \approx -\sigma \left(\frac{c}{\sigma^k} - \cdots\right).$$

The right-hand side converges to zero as σ tends to ∞. This proves the claim.

In view of the claim, to prove (6.8) it remains to show that

$$\lim_{\sigma \to \infty} \frac{\sigma}{2} \ln \left|[S_{ap}(\sigma)^{-1}]\right| = \sum \text{Re} p_i. \tag{6.10}$$

Now

$$\ln(S_{ap}(\sigma)^{-1}) = \ln \prod_i \frac{\sigma + \overline{p}_i}{\sigma - p_i} = \sum_i \ln \frac{\sigma + \overline{p}_i}{\sigma - p_i},$$

so to prove (6.10) it suffices to prove

$$\lim_{\sigma \to \infty} \frac{\sigma}{2} \ln \left|\frac{\sigma + \overline{p}_i}{\sigma - p_i}\right| = \text{Re} p_i. \tag{6.11}$$

Let $p_i = x + jy$ and use (6.9) again as follows:

$$\begin{aligned}
\frac{\sigma}{2} \ln \left|\frac{\sigma + \overline{p}_i}{\sigma - p_i}\right| &= \frac{\sigma}{2} \ln \left|\frac{1 + \overline{p}_i \sigma^{-1}}{1 - p_i \sigma^{-1}}\right| \\
&= \frac{\sigma}{4} \ln \frac{(1 + x\sigma^{-1})^2 + (y\sigma^{-1})^2}{(1 - x\sigma^{-1})^2 + (y\sigma^{-1})^2} \\
&= \frac{\sigma}{4} \left\{\ln[(1 + x\sigma^{-1})^2 + (y\sigma^{-1})^2] - \ln[(1 - x\sigma^{-1})^2 + (y\sigma^{-1})^2]\right\} \\
&= \frac{\sigma}{4} \left\{2\frac{x}{\sigma} + 2\frac{x}{\sigma} + \cdots\right\} \\
&= x + \cdots \\
&= \text{Re} p_i + \cdots.
\end{aligned}$$

Letting $\sigma \to \infty$ gives (6.11). ■

Example Take the plant and controller

$$P(s) = \frac{1}{(s-1)(s+2)}, \quad C(s) = 10.$$

The feedback system is internally stable and L has relative degree 2. The plot of $|S(j\omega)|$, log scale, versus ω, linear scale, is shown in Figure 6.1. The area below the line $|S| = 1$ is negative, the area above, positive. The theorem says that the net area is positive, equaling

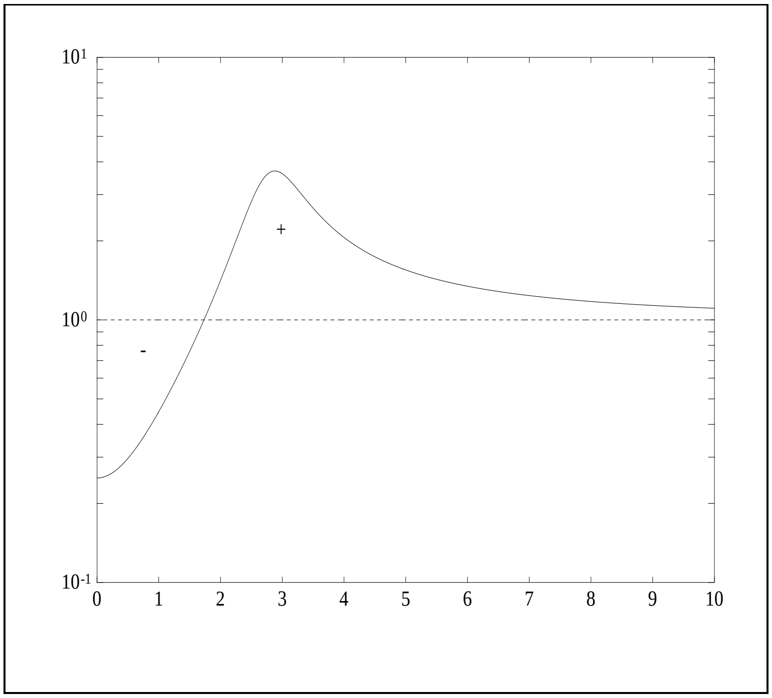

Figure 6.1: $|S(j\omega)|$, log scale, versus ω, linear scale.

$$\pi(\log \mathrm{e})(\sum \mathrm{Re} p_i) = \pi(\log \mathrm{e}).$$

So the negative area, required for good tracking over some frequency range, must unavoidably be accompanied by some positive area.

The waterbed effect applies to non-minimum-phase systems, whereas the area formula applies in general (except for the relative degree assumption). In particular, the area formula does not itself imply a peaking phenomenon, only an area conservation. However, one can infer a type of peaking phenomenon from the area formula when another constraint is imposed, namely, controller bandwidth, or more precisely, the bandwidth of the loop transfer function PC. For example, suppose that the constraint is

$$|PC| < \frac{1}{\omega^2}, \quad \forall \omega \geq \omega_1,$$

where $\omega_1 > 1$. This is one way of saying that the loop bandwidth is constrained to be $\leq \omega_1$. Then for $\omega \geq \omega_1$

$$|S| \leq \frac{1}{1 - |PC|} < \frac{1}{1 - \omega^{-2}} = \frac{\omega^2}{\omega^2 - 1}.$$

Hence

$$\int_{\omega_1}^{\infty} \log |S(j\omega)| d\omega \le \int_{\omega_1}^{\infty} \log \frac{\omega^2}{\omega^2 - 1} d\omega.$$

The latter integral—denote it by I—is finite. This is proved by the following computation:

$$\begin{aligned}
I &= \frac{1}{\ln 10} \int_{\omega_1}^{\infty} \ln \frac{1}{1 - \omega^{-2}} d\omega \\
&= -\frac{1}{\ln 10} \int_{\omega_1}^{\infty} \ln(1 - \omega^{-2}) d\omega \\
&= \frac{1}{\ln 10} \int_{\omega_1}^{\infty} \left(\omega^{-2} + \frac{1}{2}\omega^{-4} + \frac{1}{3}\omega^{-6} + \cdots \right) d\omega \\
&= \frac{1}{\ln 10} \left(\omega_1^{-1} + \frac{1}{2 \times 3}\omega_1^{-3} + \frac{1}{3 \times 5}\omega_1^{-5} + \cdots \right) \\
&< \infty.
\end{aligned}$$

Hence the possible positive area over the interval $[\omega_1, \infty)$ is limited. Thus if $|S|$ is made smaller and smaller over some subinterval of $[0, \omega_1]$, incurring a larger and larger debt of negative area, then $|S|$ must necessarily become larger and larger somewhere else in $[0, \omega_1]$. Roughly speaking, with a loop bandwidth constraint the waterbed effect applies even to minimum-phase plants.

Exercises

1. Prove the statement about uniqueness in Lemma 2.

2. True or false: For every $\delta > 1$ there exists an internally stabilizing controller such that $\|T\|_\infty < \delta$.

3. Regarding inequality (6.3), the implication is that good tracking is impossible if P has a right half-plane zero where $|W_1|$ is not small. This problem is an attempt to see this phenomenon more precisely by studying $|W_1(z)|$ as a function of z for a typical weighting function. Take W_1 to be a third-order Butterworth filter with cutoff frequency 1 rad/s. Plot

$$|W_1(0.1 + j\omega)| \quad \text{versus} \quad \omega$$

for ω going from 0 up to where $|W_1| < 0.01$. Repeat for abscissae of 1 and 10.

4. Let

$$P(s) = 4\frac{s-2}{(s+1)^2}.$$

Suppose that C is an internally stabilizing controller such that

$$\|S\|_\infty = 1.5.$$

Give a positive lower bound for

$$\max_{0 \le \omega \le 0.1} |S(j\omega)|.$$

5. Define $\epsilon := \|W_1 S\|_\infty$ and $\delta := \|CS\|_\infty$. So ϵ is a measure of tracking performance, while δ measures control effort; note that CS equals the transfer function from reference input r to plant input. In a design we would like $\epsilon < 1$ and δ not too large. Derive the following inequality, showing that ϵ and δ cannot both be very small in general: For every point s_0 with $\text{Re} s_0 \geq 0$,
$$|W_1(s_0)| \leq \epsilon + |W_1(s_0)P(s_0)|\delta.$$

6. Let ω be a frequency such that $j\omega$ is not a pole of P. Suppose that
$$\epsilon := |S(j\omega)| < 1.$$
Derive a lower bound for $|C(j\omega)|$ that blows up as $\epsilon \to 0$. Conclusion: Good tracking at a particular frequency requires large controller gain at this frequency.

7. Suppose that the plant transfer function is
$$P(s) = \frac{1}{s^2 - s + 4}.$$
We want the controller C to achieve the following:

   internal stability,
   $|S(j\omega)| \leq \epsilon$ for $0 \leq \omega < 0.1$,
   $|S(j\omega)| \leq 2$ for $0.1 \leq \omega < 5$,
   $|S(j\omega)| \leq 1$ for $5 \leq \omega < \infty$.

   Find a (positive) lower bound on the achievable ϵ.

Notes and References

This chapter is in the spirit of Bode's book (Bode, 1945) on feedback amplifiers. Bode showed that electronic amplifiers must have certain inherent properties simply by virtue of the fact that stable network functions are analytic, and hence have certain strong properties. Bode's work was generalized to control systems by Bower and Schultheiss (1961) and Horowitz (1963).

The interpolation conditions (6.1) and (6.2) were obtained by Raggazini and Franklin (1958). These constraints on S and T are essentially equivalent to the controller parametrization in Theorem 5.2. Inequality (6.3) was noted, for example, by Zames and Francis (1983). The waterbed effect, Theorem 1, was proved by Francis and Zames (1984), but the derivation here is due to Freudenberg and Looze (1985). The area formula, Theorem 2, was proved by Bode (1945) in case L is stable, and by Freudenberg and Looze (1985) in the general case. An excellent discussion of performance limitations may be found in Freudenberg and Looze (1988).