Chapter 4

Uncertainty and Robustness

No mathematical system can exactly model a physical system. For this reason we must be aware of how modeling errors might adversely affect the performance of a control system. This chapter begins with a treatment of various models of plant uncertainty. Then robust stability, stability in the face of plant uncertainty, is studied using the small-gain theorem. The final topic is robust performance, guaranteed tracking in the face of plant uncertainty.

4.1 Plant Uncertainty

The basic technique is to model the plant as belonging to a set $\mathcal{P}$. The reasons for doing this were presented in Chapter 1. Such a set can be either *structured* or *unstructured*.

For an example of a structured set consider the plant model

$$\frac{1}{s^2 + as + 1}.$$

This is a standard second-order transfer function with natural frequency 1 rad/s and damping ratio $a/2$—it could represent, for example, a mass-spring-damper setup or an R-L-C circuit. Suppose that the constant a is known only to the extent that it lies in some interval $[a_{\min}, a_{\max}]$. Then the plant belongs to the structured set

$$\mathcal{P} = \left\{ \frac{1}{s^2 + as + 1} : a_{\min} \leq a \leq a_{\max} \right\}.$$

Thus one type of structured set is parametrized by a finite number of scalar parameters (one parameter, a, in this example). Another type of structured uncertainty is a discrete set of plants, not necessarily parametrized explicitly.

For us, unstructured sets are more important, for two reasons. First, we believe that all models used in feedback design should include some unstructured uncertainty to cover unmodeled dynamics, particularly at high frequency. Other types of uncertainty, though important, may or may not arise naturally in a given problem. Second, for a specific type of unstructured uncertainty, disk uncertainty, we can develop simple, general analysis methods. Thus the basic starting point for an unstructured set is that of disk-like uncertainty. In what follows, multiplicative disk uncertainty is chosen for detailed study. This is only one type of unstructured perturbation. The important point is that we use disk uncertainty instead of a more complicated description. We do this because it greatly simplifies our analysis and lets us say some fairly precise things. The price we pay is conservativeness.

Multiplicative Perturbation

Suppose that the nominal plant transfer function is P and consider perturbed plant transfer functions of the form $\tilde{P} = (1 + \Delta W_2)P$. Here W_2 is a fixed stable transfer function, the weight, and Δ is a variable stable transfer function satisfying $\|\Delta\|_\infty < 1$. Furthermore, it is assumed that no unstable poles of P are canceled in forming $\tilde{P}$. (Thus, P and $\tilde{P}$ have the same unstable poles.) Such a perturbation Δ is said to be *allowable*.

The idea behind this uncertainty model is that ΔW_2 is the normalized plant perturbation away from 1:

$$\frac{\tilde{P}}{P} - 1 = \Delta W_2.$$

Hence if $\|\Delta\|_\infty \leq 1$, then

$$\left| \frac{\tilde{P}(j\omega)}{P(j\omega)} - 1 \right| \leq |W_2(j\omega)|, \qquad \forall \omega,$$

so $|W_2(j\omega)|$ provides the uncertainty profile. This inequality describes a disk in the complex plane: At each frequency the point $\tilde{P}/P$ lies in the disk with center 1, radius $|W_2|$. Typically, $|W_2(j\omega)|$ is a (roughly) increasing function of ω: Uncertainty increases with increasing frequency. The main purpose of Δ is to account for phase uncertainty and to act as a scaling factor on the magnitude of the perturbation (i.e., $|\Delta|$ varies between 0 and 1).

Thus, this uncertainty model is characterized by a nominal plant P together with a weighting function W_2. How does one get the weighting function W_2 in practice? This is illustrated by a few examples.

Example 1 Suppose that the plant is stable and its transfer function is arrived at by means of frequency-response experiments: Magnitude and phase are measured at a number of frequencies, $\omega_i, i = 1, \ldots, m$, and this experiment is repeated several, say n, times. Let the magnitude-phase measurement for frequency ω_i and experiment k be denoted (M_{ik}, ϕ_{ik}). Based on these data select nominal magnitude-phase pairs (M_i, ϕ_i) for each frequency ω_i, and fit a nominal transfer function $P(s)$ to these data. Then fit a weighting function $W_2(s)$ so that

$$\left| \frac{M_{ik} e^{j\phi_{ik}}}{M_i e^{j\phi_i}} - 1 \right| \leq |W_2(j\omega_i)|, \quad i = 1, \ldots, m; \; k = 1, \ldots, n.$$

Example 2 Assume that the nominal plant transfer function is a double integrator:

$$P(s) = \frac{1}{s^2}.$$

For example, a dc motor with negligible viscous damping could have such a transfer function. You can think of other physical systems with only inertia, no damping. Suppose that a more detailed model has a time delay, yielding the transfer function

$$\tilde{P}(s) = e^{-\tau s} \frac{1}{s^2},$$

and suppose that the time delay is known only to the extent that it lies in the interval $0 \leq \tau \leq 0.1$. This time-delay factor $\exp(-\tau s)$ can be treated as a multiplicative perturbation of the nominal plant by embedding $\tilde{P}$ in the family

$$\{(1 + \Delta W_2)P : \|\Delta\|_\infty \leq 1\}.$$

To do this, the weight W_2 should be chosen so that the normalized perturbation satisfies

$$\left|\frac{\tilde{P}(j\omega)}{P(j\omega)} - 1\right| \le |W_2(j\omega)|, \quad \forall \omega, \tau,$$

that is,

$$\left|e^{-\tau j\omega} - 1\right| \le |W_2(j\omega)|, \quad \forall \omega, \tau.$$

A little time with Bode magnitude plots shows that a suitable first-order weight is

$$W_2(s) = \frac{0.21s}{0.1s + 1}.$$

Figure 4.1 is the Bode magnitude plot of this W_2 and $\exp(-\tau s) - 1$ for $\tau = 0.1$, the worst value.

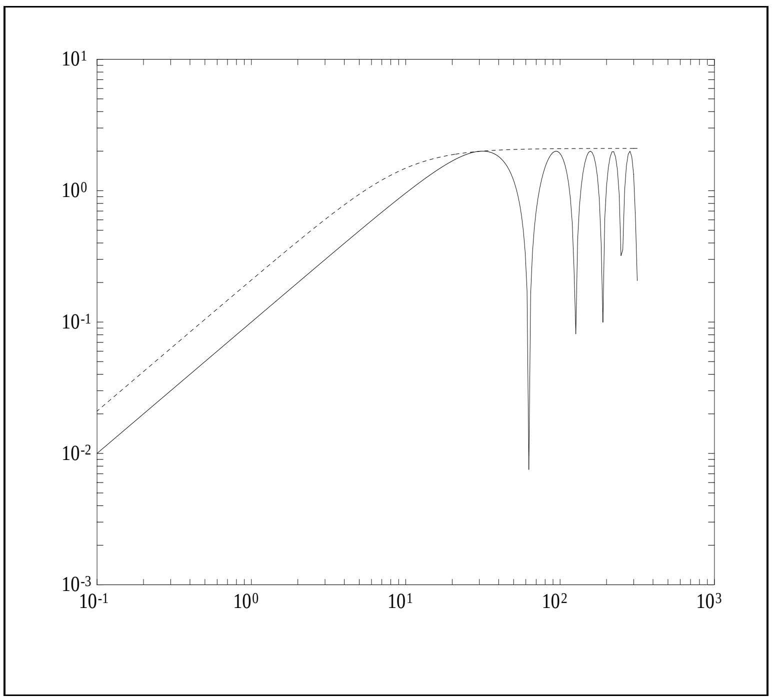

Figure 4.1: Bode plots of W_2 (dash) and $\exp(-0.1s) - 1$ (solid).

To get a feeling for how conservative this is, compare at a few frequencies ω the actual uncertainty set

$$\left\{\frac{\tilde{P}(j\omega)}{P(j\omega)} : 0 \le \tau \le 0.1\right\} = \left\{e^{-\tau j\omega} : 0 \le \tau \le 0.1\right\}$$

with the covering disk

$$\{s : |s - 1| \le |W_2(j\omega)|\}.$$

Example 3 Suppose that the plant transfer function is

$$\tilde{P}(s) = \frac{k}{s - 2},$$

where the gain k is uncertain but is known to lie in the interval $[0.1, 10]$. This plant too can be embedded in a family consisting of multiplicative perturbations of a nominal plant

$$P(s) = \frac{k_0}{s-2}.$$

The weight W_2 must satisfy

$$\left| \frac{\tilde{P}(j\omega)}{P(j\omega)} - 1 \right| \le |W_2(j\omega)|, \quad \forall \omega, k,$$

that is,

$$\max_{0.1 \le k \le 10} \left| \frac{k}{k_0} - 1 \right| \le |W_2(j\omega)|, \quad \forall \omega.$$

The left-hand side is minimized by $k_0 = 5.05$, for which the left-hand side equals $4.95/5.05$. In this way we get the nominal plant

$$P(s) = \frac{5.05}{s-2}$$

and constant weight $W_2(s) = 4.95/5.05$.

The multiplicative perturbation model is not suitable for every application because the disk covering the uncertainty set is sometimes too coarse an approximation. In this case a controller designed for the multiplicative uncertainty model would probably be too conservative for the original uncertainty model.

The discussion above illustrates an important point. In modeling a plant we may arrive at a certain plant set. This set may be too awkward to cope with mathematically, so we may embed it in a larger set that is easier to handle. Conceivably, the achievable performance for the larger set may not be as good as the achievable performance for the smaller; that is, there may exist—even though we cannot find it—a controller that is better for the smaller set than the controller we design for the larger set. In this sense the latter controller is *conservative* for the smaller set.

In this book we stick with plant uncertainty that is disk-like. It will be conservative for some problems, but the payoff is that we obtain some very nice theoretical results. The resulting theory is remarkably practical as well.

Other Perturbations

Other uncertainty models are possible besides multiplicative perturbations, as illustrated by the following example.

Example 4 As at the start of this section, consider the family of plant transfer functions

$$\frac{1}{s^2 + as + 1}, \quad 0.4 \le a \le 0.8.$$

Thus

$$a = 0.6 + 0.2\Delta, \quad -1 \le \Delta \le 1,$$

so the family can be expressed as

$$\frac{P(s)}{1 + \Delta W_2(s) P(s)}, \quad -1 \le \Delta \le 1,$$

where

$$P(s) := \frac{1}{s^2 + 0.6s + 1}, \quad W_2(s) := 0.2s.$$

Note that P is the nominal plant transfer function for the value $a = 0.6$, the midpoint of the interval. The block diagram corresponding to this representation of the plant is shown in Figure 4.2. Thus

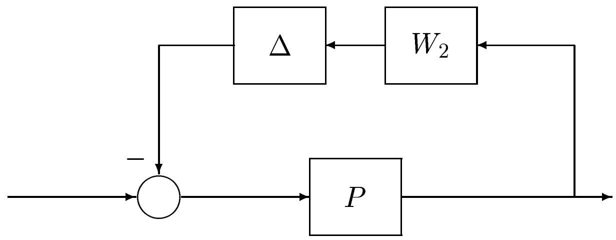

Figure 4.2: Example 4.

the original plant has been represented as a feedback uncertainty around a nominal plant.

The following list summarizes the common uncertainty models:

$$\begin{gathered}(1 + \Delta W_2)P \\ P + \Delta W_2 \\ P/(1 + \Delta W_2 P) \\ P/(1 + \Delta W_2)\end{gathered}$$

Appropriate assumptions would be made on Δ and W_2 in each case. Typically, we can relax the assumption that Δ be stable; but then the theorems to follow would be harder to prove.

4.2 Robust Stability

The notion of robustness can be described as follows. Suppose that the plant transfer function P belongs to a set $\mathcal{P}$, as in the preceding section. Consider some characteristic of the feedback system, for example, that it is internally stable. A controller C is *robust* with respect to this characteristic if this characteristic holds for every plant in $\mathcal{P}$. The notion of robustness therefore requires a controller, a set of plants, and some characteristic of the system. For us, the two most important variations of this notion are robust stability, treated in this section, and robust performance, treated in the next.

A controller C provides *robust stability* if it provides internal stability for every plant in $\mathcal{P}$. We might like to have a test for robust stability, a test involving C and $\mathcal{P}$. Or if $\mathcal{P}$ has an associated size, the maximum size such that C stabilizes all of $\mathcal{P}$ might be a useful notion of stability margin.

The Nyquist plot gives information about stability margin. Note that the distance from the critical point -1 to the nearest point on the Nyquist plot of L equals $1/\|S\|_\infty$:

$$\begin{aligned}\text{distance from -1 to Nyquist plot} &= \inf_\omega |-1 - L(j\omega)| \\ &= \inf_\omega |1 + L(j\omega)| \\ &= \left[\sup_\omega \frac{1}{|1 + L(j\omega)|}\right]^{-1}\end{aligned}$$

$$= \quad \|S\|_\infty^{-1}.$$

Thus if $\|S\|_\infty \gg 1$, the Nyquist plot comes close to the critical point, and the feedback system is nearly unstable. However, as a measure of stability margin this distance is not entirely adequate because it contains no frequency information. More precisely, if the nominal plant P is perturbed to $\tilde{P}$, having the same number of unstable poles as has P and satisfying the inequality

$$|\tilde{P}(j\omega)C(j\omega) - P(j\omega)C(j\omega)| < \|S\|_\infty^{-1}, \quad \forall\omega,$$

then internal stability is preserved (the number of encirclements of the critical point by the Nyquist plot does not change). But this is usually very conservative; for instance, larger perturbations could be allowed at frequencies where $P(j\omega)C(j\omega)$ is far from the critical point.

Better stability margins are obtained by taking explicit frequency-dependent perturbation models: for example, the multiplicative perturbation model, $\tilde{P} = (1 + \Delta W_2)P$. Fix a positive number β and consider the family of plants

$$\{\tilde{P} : \Delta \text{ is stable and } \|\Delta\|_\infty \le \beta\}.$$

Now a controller C that achieves internal stability for the nominal plant P will stabilize this entire family if β is small enough. Denote by β_{sup} the least upper bound on β such that C achieves internal stability for the entire family. Then β_{sup} is a stability margin (with respect to this uncertainty model). Analogous stability margins could be defined for the other uncertainty models.

We turn now to two classical measures of stability margin, gain and phase margin. Assume that the feedback system is internally stable with plant P and controller C. Now perturb the plant to kP, with k a positive real number. The *upper gain margin*, denoted $k_{\max}$, is the first value of k greater than 1 when the feedback system is not internally stable; that is, $k_{\max}$ is the maximum number such that internal stability holds for $1 \le k < k_{\max}$. If there is no such number, then set $k_{\max} := \infty$. Similarly, the *lower gain margin*, $k_{\min}$, is the least nonnegative number such that internal stability holds for $k_{\min} < k \le 1$. These two numbers can be read off the Nyquist plot of L; for example, $-1/k_{\max}$ is the point where the Nyquist plot intersects the segment $(-1, 0)$ of the real axis, the closest point to -1 if there are several points of intersection.

Now perturb the plant to $\mathrm{e}^{-j\phi}P$, with ϕ a positive real number. The *phase margin*, $\phi_{\max}$, is the maximum number (usually expressed in degrees) such that internal stability holds for $0 \le \phi < \phi_{\max}$. You can see that $\phi_{\max}$ is the angle through which the Nyquist plot must be rotated until it passes through the critical point, -1; or, in radians, $\phi_{\max}$ equals the arc length along the unit circle from the Nyquist plot to the critical point.

Thus gain and phase margins measure the distance from the critical point to the Nyquist plot in certain specific directions. Gain and phase margins have traditionally been important measures of stability robustness: if either is small, the system is close to instability. Notice, however, that the gain and phase margins can be relatively large and yet the Nyquist plot of L can pass close to the critical point; that is, *simultaneous* small changes in gain and phase could cause internal instability. We return to these margins in Chapter 11.

Now we look at a typical robust stability test, one for the multiplicative perturbation model. Assume that the nominal feedback system (i.e., with $\Delta = 0$) is internally stable for controller C. Bring in again the complementary sensitivity function

$$T = 1 - S = \frac{L}{1+L} = \frac{PC}{1+PC}.$$

Theorem 1 *(Multiplicative uncertainty model) C provides robust stability iff $\|W_2 T\|_\infty < 1$.*

Proof ($\Leftarrow$) Assume that $\|W_2T\|_\infty < 1$. Construct the Nyquist plot of L, indenting $\mathcal{D}$ to the left around poles on the imaginary axis. Since the nominal feedback system is internally stable, we know this from the Nyquist criterion: The Nyquist plot of L does not pass through -1 and its number of counterclockwise encirclements equals the number of poles of P in Re$s \geq 0$ plus the number of poles of C in Re$s \geq 0$.

Fix an allowable Δ. Construct the Nyquist plot of $\tilde{P}C = (1 + \Delta W_2)L$. No additional indentations are required since ΔW_2 introduces no additional imaginary axis poles. We have to show that the Nyquist plot of $(1 + \Delta W_2)L$ does not pass through -1 and its number of counterclockwise encirclements equals the number of poles of $(1 + \Delta W_2)P$ in Re $s \geq 0$ plus the number of poles of C in Re $s \geq 0$; equivalently, the Nyquist plot of $(1 + \Delta W_2)L$ does not pass through -1 and encircles it exactly as many times as does the Nyquist plot of L. We must show, in other words, that the perturbation does not change the number of encirclements.

The key equation is

$$1 + (1 + \Delta W_2)L = (1 + L)(1 + \Delta W_2 T). \tag{4.1}$$

Since

$$\|\Delta W_2 T\|_\infty \leq \|W_2 T\|_\infty < 1,$$

the point $1 + \Delta W_2 T$ always lies in some closed disk with center 1, radius < 1, for all points s on $\mathcal{D}$. Thus from (4.1), as s goes once around $\mathcal{D}$, the net change in the angle of $1 + (1 + \Delta W_2)L$ equals the net change in the angle of $1 + L$. This gives the desired result.

($\Rightarrow$) Suppose that $\|W_2T\|_\infty \geq 1$. We will construct an allowable Δ that destabilizes the feedback system. Since T is strictly proper, at some frequency ω,

$$|W_2(j\omega)T(j\omega)| = 1. \tag{4.2}$$

Suppose that $\omega = 0$. Then $W_2(0)T(0)$ is a real number, either $+1$ or -1. If $\Delta = -W_2(0)T(0)$, then Δ is allowable and

$$1 + \Delta W_2(0)T(0) = 0.$$

From (4.1) the Nyquist plot of $(1 + \Delta W_2)L$ passes through the critical point, so the perturbed feedback system is not internally stable.

If $\omega > 0$, constructing an admissible Δ takes a little more work; the details are omitted. ■

The theorem can be used effectively to find the stability margin β_{sup} defined previously. The simple scaling technique

$$\begin{aligned}\{\tilde{P} = (1 + \Delta W_2)P : \|\Delta\|_\infty \leq \beta\} &= \{\tilde{P} = (1 + \beta^{-1}\Delta\beta W_2)P : \|\beta^{-1}\Delta\|_\infty \leq 1\} \\ &= \{\tilde{P} = (1 + \Delta_1 \beta W_2)P : \|\Delta_1\|_\infty \leq 1\}\end{aligned}$$

together with the theorem shows that

$$\beta_{\text{sup}} = \sup\{\beta : \|\beta W_2 T\|_\infty < 1\} = 1/\|W_2T\|_\infty.$$

The condition $\|W_2T\|_\infty < 1$ also has a nice graphical interpretation. Note that

$$\begin{aligned}\|W_2T\|_\infty < 1 \quad &\Leftrightarrow \quad \left|\frac{W_2(j\omega)L(j\omega)}{1 + L(j\omega)}\right| < 1, \quad \forall\omega \\ &\Leftrightarrow \quad |W_2(j\omega)L(j\omega)| < |1 + L(j\omega)|, \quad \forall\omega.\end{aligned}$$

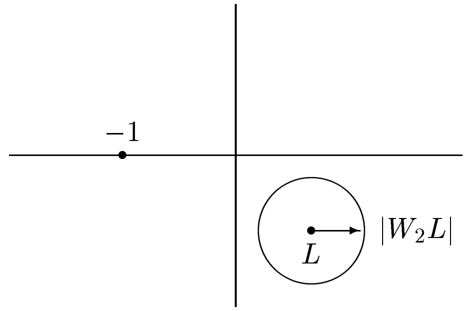

Figure 4.3: Robust stability graphically.

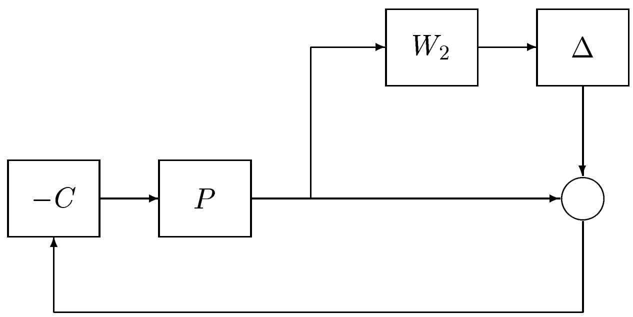

Figure 4.4: Perturbed feedback system.

The last inequality says that at every frequency, the critical point, -1, lies outside the disk of center $L(j\omega)$, radius $|W_2(j\omega)L(j\omega)|$ (Figure 4.3).

There is a simple way to see the relevance of the condition $\|W_2T\|_\infty < 1$. First, draw the block diagram of the perturbed feedback system, but ignoring inputs (Figure 4.4). The transfer function from the output of Δ around to the input of Δ equals $-W_2T$, so the block diagram collapses to the configuration shown in Figure 4.5. The maximum loop gain in Figure 4.5 equals $\|-\Delta W_2T\|_\infty$,

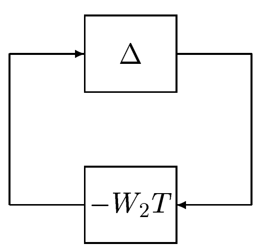

Figure 4.5: Collapsed block diagram.

which is < 1 for all allowable Δs iff the small-gain condition $\|W_2T\|_\infty < 1$ holds.

The foregoing discussion is related to the *small-gain theorem*, a special case of which is this: If L is stable and $\|L\|_\infty < 1$, then $(1 + L)^{-1}$ is stable too. An easy proof uses the Nyquist criterion.

Summary of Robust Stability Tests

Table 4.1 summarizes the robust stability tests for the other uncertainty models.

Perturbation	Condition
$(1+\Delta W_2)P$	$\|W_2T\|_\infty < 1$
$P+\Delta W_2$	$\|W_2CS\|_\infty < 1$
$P/(1+\Delta W_2P)$	$\|W_2PS\|_\infty < 1$
$P/(1+\Delta W_2)$	$\|W_2S\|_\infty < 1$

Table 4.1

Note that we get the same four transfer functions—T, CS, PS, S—as we did in Section 3.4. This should not be too surprising since (up to sign) these are the only closed-loop transfer functions for a unity feedback SISO system.

4.3 Robust Performance

Now we look into performance of the perturbed plant. Suppose that the plant transfer function belongs to a set $\mathcal{P}$. The general notion of *robust performance* is that internal stability and performance, of a specified type, should hold for all plants in $\mathcal{P}$. Again we focus on multiplicative perturbations.

Recall that when the nominal feedback system is internally stable, the *nominal performance* condition is $\|W_1S\|_\infty < 1$ and the *robust stability* condition is $\|W_2T\|_\infty < 1$. If P is perturbed to $(1+\Delta W_2)P$, S is perturbed to

$$\frac{1}{1+(1+\Delta W_2)L} = \frac{S}{1+\Delta W_2 T}.$$

Clearly, the *robust performance* condition should therefore be

$$\|W_2T\|_\infty < 1 \text{ and } \left\| \frac{W_1S}{1+\Delta W_2T} \right\|_\infty < 1, \quad \forall \Delta.$$

Here Δ must be allowable. The next theorem gives a test for robust performance in terms of the function

$$s \mapsto |W_1(s)S(s)| + |W_2(s)T(s)|,$$

which is denoted $|W_1S| + |W_2T|$.

Theorem 2 *A necessary and sufficient condition for robust performance is*

$$\| |W_1S| + |W_2T| \|_\infty < 1. \tag{4.3}$$

Proof ($\Leftarrow$) Assume (4.3), or equivalently,

$$\|W_2T\|_\infty \text{ and } \left\| \frac{W_1S}{1-|W_2T|} \right\|_\infty < 1. \tag{4.4}$$

Fix Δ. In what follows, functions are evaluated at an arbitrary point $j\omega$, but this is suppressed to simplify notation. We have

$$1 = |1 + \Delta W_2 T - \Delta W_2 T| \le |1 + \Delta W_2 T| + |W_2 T|$$

and therefore

$$1 - |W_2 T| \le |1 + \Delta W_2 T|.$$

This implies that

$$\left\| \frac{W_1 S}{1 - |W_2 T|} \right\|_\infty \ge \left\| \frac{W_1 S}{1 + \Delta W_2 T} \right\|_\infty .$$

This and (4.4) yield

$$\left\| \frac{W_1 S}{1 + \Delta W_2 T} \right\|_\infty < 1.$$

($\Rightarrow$) Assume that

$$\|W_2 T\|_\infty < 1 \text{ and } \left\| \frac{W_1 S}{1 + \Delta W_2 T} \right\|_\infty < 1, \quad \forall \Delta. \tag{4.5}$$

Pick a frequency ω where

$$\frac{|W_1 S|}{1 - |W_2 T|}$$

is maximum. Now pick Δ so that

$$1 - |W_2 T| = |1 + \Delta W_2 T|.$$

The idea here is that $\Delta(j\omega)$ should rotate $W_2(j\omega)T(j\omega)$ so that $\Delta(j\omega)W_2(j\omega)T(j\omega)$ is negative real. The details of how to construct such an allowable Δ are omitted. Now we have

$$\begin{aligned} \left\| \frac{W_1 S}{1 - |W_2 T|} \right\|_\infty &= \frac{|W_1 S|}{1 - |W_2 T|} \\ &= \frac{|W_1 S|}{|1 + \Delta W_2 T|} \\ &\le \left\| \frac{W_1 S}{1 + \Delta W_2 T} \right\|_\infty . \end{aligned}$$

So from this and (4.5) there follows (4.4). ■

Test (4.3) also has a nice graphical interpretation. For each frequency ω, construct two closed disks: one with center -1, radius $|W_1(j\omega)|$; the other with center $L(j\omega)$, radius $|W_2(j\omega)L(j\omega)|$. Then (4.3) holds iff for each ω these two disks are disjoint (Figure 4.6).

The robust performance condition says that the robust performance level 1 is achieved. More generally, let's say that robust performance level α is achieved if

$$\|W_2 T\|_\infty < 1 \text{ and } \left\| \frac{W_1 S}{1 + \Delta W_2 T} \right\|_\infty < \alpha, \quad \forall \Delta.$$

Noting that at every frequency

$$\max_{|\Delta| \le 1} \left| \frac{W_1 S}{1 + \Delta W_2 T} \right| = \frac{|W_1 S|}{1 - |W_2 T|}$$

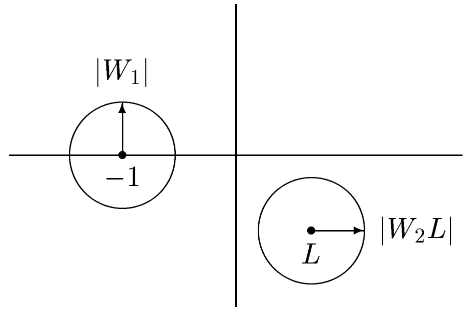

Figure 4.6: Robust performance graphically.

we get that the minimum α equals

$$\left\| \frac{W_1 S}{1 - |W_2 T|} \right\|_\infty . \tag{4.6}$$

Alternatively, we may wish to know how large the uncertainty can be while the robust performance condition holds. To do this, we scale the uncertainty level, that is, we allow Δ to satisfy $\|\Delta\|_\infty < \beta$. Application of Theorem 1 shows that internal stability is robust iff $\|\beta W_2 T\|_\infty < 1$. Let's say that the uncertainty level β is permissible if

$$\|\beta W_2 T\|_\infty < 1 \text{ and } \left\| \frac{W_1 S}{1 + \Delta W_2 T} \right\|_\infty < 1, \quad \forall \Delta.$$

Again, noting that

$$\max_{|\Delta| \le 1} \left| \frac{W_1 S}{1 + \beta \Delta W_2 T} \right| = \frac{|W_1 S|}{1 - \beta |W_2 T|},$$

we get that the maximum β equals

$$\left\| \frac{W_2 T}{1 - |W_1 S|} \right\|_\infty^{-1} .$$

Now we turn briefly to some related problems.

Robust Stability for Multiple Perturbations

Suppose that a nominal plant P is perturbed to

$$\tilde{P} = P \frac{1 + \Delta_2 W_2}{1 + \Delta_1 W_1}$$

with W_1, W_2 both stable and Δ_1, Δ_2 both admissible. The robust stability condition is

$$\| |W_1 S| + |W_2 T| \|_\infty < 1,$$

which is just the robust performance condition in Theorem 2. A sketch of the proof goes like this: From the fourth entry in Table 4.1, for fixed Δ_2 the robust stability condition for varying Δ_1 is

$$\left\| W_1 \frac{1}{1 + (1 + \Delta_2 W_2) L} \right\|_\infty < 1.$$

Then from Theorem 2 this holds for all admissible Δ_2 iff

$$\| |W_1 S| + |W_2 T| \|_\infty < 1.$$

This illustrates a more general point: Robust performance with one perturbation is equivalent to robust stability with two perturbations, provided that performance is in terms of the ∞-norm and the second perturbation is chosen appropriately.

Robust Command Response

Consider the block diagram shown in Figure 4.7. Shown are a plant P and two controller compo-

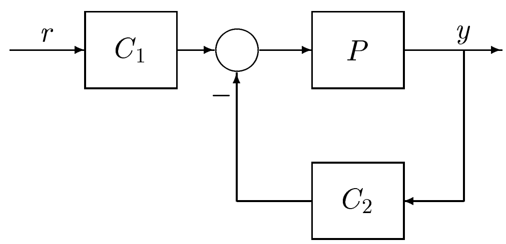

Figure 4.7: Two-degree-of-freedom controller.

nents, C_1 and C_2. This is known as a two-degree-of-freedom controller because the plant input is allowed to be a function of the two signals r and y independently, not just $r - y$. We will not go into details about such controllers or about the appropriate definition of internal stability.

Define

$$S := \frac{1}{1 + PC_2}, \quad T := 1 - S.$$

Then the transfer function from r to y, denoted T_{yr}, is

$$T_{yr} = PSC_1.$$

Let M be a transfer function representing a model that we want the foregoing system to emulate. Denote by e the difference between y and the output of M. The error transfer function, that from r to e, is

$$T_{er} = T_{yr} - M = PSC_1 - M.$$

The ideal choice for C_1, the one making $T_{er} = 0$, would therefore be

$$C_1 = \frac{M}{PS}.$$

This choice may violate the internal stability constraint, but let's suppose that in order to continue that it does not (this places some limitations on M).

Consider now a multiplicative perturbation of the plant: P becomes $\tilde{P} = (1 + \Delta W_2)P$, Δ admissible. Then T_{er} becomes

$$\begin{aligned}
\tilde{T}_{er} &= \frac{\tilde{P}C_1}{1 + \tilde{P}C_2} - M \\
&= \frac{\tilde{P}}{1 + \tilde{P}C_2}\frac{M}{PS} - M \\
&= \frac{\Delta W_2 MS}{1 + \Delta W_2 T} \text{ (after some algebra).}
\end{aligned}$$

Defining $W_1 := W_2 M$, we find that the maximum ∞-norm of the error transfer function, over all admissible Δ, is

$$\max_{\Delta} \|\tilde{T}_{ec}\|_\infty = \left\| \frac{W_1 S}{1 - |W_2 T|} \right\|_\infty .$$

The right-hand side we have already seen in (4.6).

Note that we convert the problem of making the closed-loop response from r to y match some desired response by subtracting off that desired response and forming an error signal e which we seek to keep small. In some treatments of the command response problem, the performance specification is taken to be: make $|T_{yr}|$ close to a desired model. The problem with this specification is that two transfer functions can be close in magnitude but differ substantially in phase. Surprisingly, this can occur even when both transfer functions are minimum phase. The interested reader may want to investigate this further using the gain-phase relation developed in Chapter 7.

4.4 Robust Performance More Generally

Theorem 2 gives the robust performance test under the following conditions:

Perturbation model:	$(1 + \Delta W_2)P$
Nominal performance condition:	$\|W_1 S\|_\infty < 1$

Table 4.2 gives tests for the four uncertainty models and two nominal performance conditions.

	Nominal Performance Condition	
Perturbation	$\|W_1 S\|_\infty < 1$	$\|W_1 T\|_\infty < 1$
$(1 + \Delta W_2)P$	$\||W_1 S| + |W_2 T|\|_\infty < 1$	messy
$P + W_2 \Delta$	$\||W_1 S| + |W_2 C S|\|_\infty < 1$	messy
$P/(1 + \Delta W_2 P)$	messy	$\||W_1 T| + |W_2 P S|\|_\infty < 1$
$P/(1 + \Delta W_2)$	messy	$\||W_1 T| + |W_2 S|\|_\infty < 1$

Table 4.2

The entries marked *messy* are just that. The difficulty is the way in which Δ enters. For example, consider the case where

Perturbation model:	$(1 + \Delta W_2)P$
Nominal performance condition:	$\|W_1 T\|_\infty < 1$

The perturbed T is

$$\frac{(1 + \Delta W_2)PC}{1 + (1 + \Delta W_2)PC} = \frac{(1 + \Delta W_2)T}{1 + \Delta W_2 T},$$

so the perturbed performance condition is equivalent to

$$|W_1(1 + \Delta W_2)T| < |1 + \Delta W_2 T|, \quad \forall \omega.$$

Now for each fixed ω

$$|W_1(1 + \Delta W_2)T| \le |W_1 T|(1 + |W_2|)$$

and

$$1 - |W_2 T| \le |1 + \Delta W_2 T|.$$

So a sufficient condition for robust performance is

$$\left\| \frac{W_1 T(1 + |W_2|)}{1 - |W_2 T|} \right\|_\infty < 1.$$

4.5 Conclusion

The nominal feedback system is assumed to be internally stable. Then the *nominal performance* condition is $\|W_1 S\|_\infty < 1$ and the *robust stability* condition (with respect to multiplicative perturbations) is $\|W_2 T\|_\infty < 1$.

The condition for simultaneously achieving nominal performance and robust stability is

$$\| \max (|W_1 S|, |W_2 T|)\|_\infty < 1. \tag{4.7}$$

The *robust performance* condition is

$$\|W_2 T\|_\infty < 1 \text{ and } \left\| \frac{W_1 S}{1 + \Delta W_2 T} \right\|_\infty < 1, \quad \forall \Delta$$

and the test for this is

$$\| |W_1 S| + |W_2 T| \|_\infty < 1. \tag{4.8}$$

Since

$$\max (|W_1 S|, |W_2 T|) \le |W_1 S| + |W_2 T| \le 2 \max (|W_1 S|, |W_2 T|) \tag{4.9}$$

conditions (4.7) and (4.8) are not too far apart. For instance, if nominal performance and robust stability are obtained with a safety factor of 2, that is,

$$\|W_1 S\|_\infty < 1/2, \quad \|W_2 T\|_\infty < 1/2,$$

then robust performance is automatically obtained.

A compromise condition, which we shall treat in Chapters 8 and 12, is

$$\|(|W_1 S|^2 + |W_2 T|^2)^{1/2}\|_\infty < 1. \tag{4.10}$$

Simple plane geometry shows that

$$\max (|W_1 S|, |W_2 T|) \le (|W_1 S|^2 + |W_2 T|^2)^{1/2} \le |W_1 S| + |W_2 T| \tag{4.11}$$

and

$$\frac{1}{\sqrt{2}}(|W_1 S| + |W_2 T|) \le (|W_1 S|^2 + |W_2 T|^2)^{1/2} \le \sqrt{2} \max (|W_1 S|, |W_2 T|). \tag{4.12}$$

Thus (4.10) is a reasonable approximation to both (4.7) and (4.8).

To elaborate on this point, let's consider

$$x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} |W_1 S| \\ |W_2 T| \end{pmatrix}$$

as a vector in $\mathbb{R}^2$. Then (4.7), (4.8), and (4.10) correspond, respectively, to the three different norms

$$\max (|x_1|, |x_2|), \quad |x_1| + |x_2|, \quad (|x_1|^2 + |x_2|^2)^{1/2}.$$

The third is the Euclidean norm and is the most tractable. The point being made here is that choice of these spatial norms is not crucial: The tradeoffs between $|W_1 S|$ and $|W_2 T|$ inherent in control problems mean that although the norms may differ by as much as a factor of 2, the actual solutions one gets by using the different norms are not very different.

Exercises

1. Consider a unity-feedback system. True or false: If a controller internally stabilizes two plants, they have the same number of poles in Re$s \geq 0$.

2. Unity-feedback problem. Let $P_\alpha(s)$ be a plant depending on a real parameter α. Suppose that the poles of P_α move continuously as α varies over the interval $[0,1]$. True or false: If a controller internally stabilizes P_α for every α in $[0,1]$, then P_α has the same number of poles in Re $s \geq 0$ for every α in $[0,1]$.

3. For the unity-feedback system with $P(s) = k/s$, does there exist a proper controller $C(s)$ such that the feedback system is internally stable for both $k = +1$ and $k = -1$?

4. Suppose that
$$P(s) = \frac{\omega_n^2}{s(s + 2\zeta\omega_n)}, \quad C(s) = 1$$
with $\omega_n, \zeta > 0$. Note that the characteristic polynomial is the standard second-order one. Find the phase margin as a function of ζ. Sketch the graph of this function.

5. Consider the unity-feedback system with
$$P(s) = \frac{1}{(s+1)(s+\alpha)}, \quad C(s) = \frac{1}{s}.$$
For what range of α (a real number) is the feedback system internally stable? Find the upper and lower gain margins as functions of α.

6. This problem studies robust stability for real parameter variations. Consider the unity-feedback system with $C(s) = 10$ and plant
$$\frac{1}{s-a},$$
where a is real.

   (a) Find the range of a for the feedback system to be internally stable.

   (b) For $a = 0$ the plant is $P(s) = 1/s$. Regarding a as a perturbation, we can write the plant as
$$\tilde{P} = \frac{P}{1 + \Delta W_2 P}$$
   with $W_2(s) = -a$. Then $\tilde{P}$ equals the true plant when $\Delta(s) = 1$. Apply robust stability theory to see when the feedback system with plant $\tilde{P}$ is internally stable for all $\|\Delta\|_\infty \leq 1$. You will get a smaller range for a than in part (a).

   (c) Repeat with the nominal plant $P(s) = 1/(s + 100)$.

7. This problem concerns robust stability of the unity-feedback system. Suppose that P and C are nominal transfer functions for which the feedback system is internally stable. Instead of allowing perturbations in just P, this problem allows perturbations in C too. Suppose that P may be perturbed to
$$(1 + \Delta_1 W)P$$

and C may be perturbed to

$$(1 + \Delta_2 V)C.$$

The transfer functions W and V are fixed, while Δ_1 and Δ_2 are variable transfer functions having ∞-norms no greater than 1. Making appropriate additional assumptions, find a sufficient condition, depending only on the four functions P, C, W, V, for robust stability. Prove sufficiency. (A weak sufficient condition is the goal; for example, the condition $W = V = 0$ would be too strong.)

8. Assume that the nominal plant transfer function is a double integrator,

$$P(s) = \frac{1}{s^2}.$$

The performance requirement is that the plant output should track reference inputs over the frequency range $[0, 1]$. The performance weight W_1 could therefore be chosen so that its magnitude is constant over this frequency range and then rolls off at higher frequencies. A common choice for W_1 is a Butterworth filter, which is maximally flat over its bandwidth. Choose a third-order Butterworth filter for W_1 with cutoff frequency 1 rad/s. Take the weight W_2 to be

$$W_2(s) = \frac{0.21s}{0.1s + 1}.$$

(a) Design a proper C to achieve internal stability for the nominal plant.

(b) Check the robust stability condition $\|W_2 T\|_\infty < 1$. If this does not hold, redesign C until it does. It is not necessary to get a C that yields good performance.

(c) Compute the robust performance level α for your controller from (4.6).

9. Consider the class of perturbed plants of the form

$$\frac{P}{1 + \Delta W_2 P},$$

where W_2 is a fixed stable weighting function with $W_2 P$ strictly proper and Δ is a variable stable transfer function with $\|\Delta\|_\infty \leq 1$. Assume that C is a controller achieving internal stability for P. Prove that C provides internal stability for the perturbed plant if $\|W_2 P S\|_\infty < 1$.

10. Suppose that the plant transfer function is

$$\tilde{P}(s) = [1 + \Delta(s) W_2(s)]\, P(s),$$

where

$$W_2(s) = \frac{2}{s + 10}, \quad P(s) = \frac{1}{s - 1},$$

and the stable perturbation Δ satisfies $\|\Delta\|_\infty \leq 2$. Suppose that the controller is the pure gain $C(s) = k$. We want the feedback system to be internally stable for all such perturbations. Determine over what range of k this is true.

Notes and References

The basis for this chapter is Doyle and Stein (1981). This paper emphasized the importance of explicit uncertainty models such as multiplicative and additive. Theorem 1 is stated in that paper, but a complete proof is due to Chen and Desoer (1982). The sufficiency part of this theorem is a version of the small-gain theorem, due to Sandberg and Zames [see, e.g., Desoer and Vidyasagar (1975)].