CALIFORNIA INSTITUTE OF TECHNOLOGY
Control and Dynamical Systems

CDS 101

D. MacMartin | **Problem Set #1** | Issued: 30 Sep 13
Fall 2013 | | Due: 9 Oct 13

Note: In the upper left hand corner of the *second* page of your homework set, please put the number of hours that you spent on this homework set (including reading).

1. Åström and Murray, Exercise 1.2

2. Consider the cruise-control example discussed in class, with

$$m\dot{v} = -av + u + w$$

where u is the control input (force applied by engine) and w the disturbance input (force applied by hill, etc.), which will be ignored below ($w = 0$). An *open-loop* control strategy to achieve a given reference speed v_{ref} would be to choose

$$u = \hat{a} v_{\text{ref}}$$

where $\hat{a}$ is your estimate of a, which may not be accurate. Assume m, a and $\hat{a}$ are all positive.

(a) Compute the steady-state response for both the open-loop strategy above, and for the feedback law

$$u = -k_p(v - v_{\text{ref}})$$

and compare the steady-state (with $w = 0$) as a function of $\beta = a/\hat{a}$ when $k_p = 10\hat{a}$. (You should solve the problem analytically, and then plot the response $v_{\text{ss}}/v_{\text{ref}}$ as a function of β for both the open-loop and proportional-gain feedback law.)

(b) Now consider a proportional-integral control law

$$u = -k_p(v - v_{\text{ref}}) - k_i \int_0^t (v - v_{\text{ref}})dt$$

and again compute the steady state solution (assuming stability) and compare the response with the proportional gain case from above. (Note that if you define $q = \int_0^t (v - v_{\text{ref}})dt$ then $\dot{q} = v - v_{\text{ref}}$.)

3. Åström and Murray, Exercise 2.6, parts (a) and (b)

CALIFORNIA INSTITUTE OF TECHNOLOGY
Control and Dynamical Systems

CDS 110a

D. MacMartin | **Problem Set #1** | Issued: 30 Sep 13
Fall 2013 | | Due: 9 Oct 13

Note: In the upper left hand corner of the *second* page of your homework set, please put the number of hours that you spent on this homework set (including reading).

1. Åström and Murray, Exercise 1.2

2. Åström and Murray, Exercise 2.1

3. Consider the cruise-control example discussed in class, with

$$m\dot{v} = -av + u + w$$

where u is the control input (force applied by engine) and w the disturbance input (force applied by hill, etc.), which will be ignored below ($w = 0$). An *open-loop* control strategy to achieve a given reference speed v_{ref} would be to choose

$$u = \hat{a}v_{\mathrm{ref}}$$

where $\hat{a}$ is your estimate of a, which may not be accurate. Assume m, a, and $\hat{a}$ are all positive.

(a) Compute the steady-state response for both the open-loop strategy above, and for the feedback law

$$u = -k_p(v - v_{\mathrm{ref}})$$

and compare the steady-state (with $w = 0$) as a function of $\beta = a/\hat{a}$ when $k_p = 10\hat{a}$. (You should solve the problem analytically, and then plot the response $v_{\mathrm{ss}}/v_{\mathrm{ref}}$ as a function of β for both the open-loop and proportional-gain feedback law.)

(b) Now consider a proportional-integral (PI) control law

$$u = -k_p(v - v_{\mathrm{ref}}) - k_i \int_0^t (v - v_{\mathrm{ref}})dt$$

and again compute the steady state solution (assuming stability) and compare the response with the proportional gain case from above. (Note that if you define $q = \int_0^t (v - v_{\mathrm{ref}})dt$ then $\dot{q} = v - v_{\mathrm{ref}}$.)

(c) Next, simulate the response of the system (using `ode45` in Matlab or `odeint` in SciPy or something similar) with the PI control law above with $m = 1$, $a = 0.1$, $w = 0$, and "input" to the system of $v_{\mathrm{ref}} = \sin(\omega t)$, for ω=0.01, 0.1, 1, and 10 rad/sec. In each case, you should simulate at least 10 cycles; after some initial transient, the response should be periodic. Compute the peak-to-peak amplitude of the final period for the error $v - v_{\mathrm{ref}}$, and plot this as a function of frequency on a log-log scale, for the following control gains:

i. $k_p = 1$, $k_i = 0$

ii. $k_p = 1$, $k_i = 1$

iii. $k_p = 1$, $k_i = 10$

(If you want to see interesting behaviour, simulate the final case at $\omega = 3.3\,\text{rad/sec}$ as well.)

4. Consider a damped spring–mass system with dynamics

$$m\ddot{q} + c\dot{q} + kq = F.$$

Let $\omega_0 = \sqrt{k/m}$ be the natural frequency and $\zeta = c/(2\sqrt{km})$ be the damping ratio.

(a) Show that by rescaling the equations, we can write the dynamics in the form

$$\ddot{q} + 2\zeta\omega_0\dot{q} + \omega_0^2 q = \omega_0^2 u, \tag{S1.1}$$

where $u = F/k$. This form of the dynamics is that of a linear oscillator with natural frequency ω_0 and damping ratio ζ.

(b) Show that the system can be further normalized (you will need to rescale the time variable as well as identifying states) and written in the form

$$\frac{dz_1}{d\tau} = z_2, \qquad \frac{dz_2}{d\tau} = -z_1 - 2\zeta z_2 + v. \tag{S1.2}$$

The essential dynamics of the system are governed by a single damping parameter ζ. The *Q-value* defined as $Q = 1/2\zeta$ is sometimes used instead of ζ.

(c) Show that the solution for the unforced system ($v = 0$) with no damping ($\zeta = 0$) is given by

$$z_1(\tau) = z_1(0)\cos\tau + z_2(0)\sin\tau, \qquad z_2(\tau) = -z_1(0)\sin\tau + z_2(0)\cos\tau.$$

Invert the scaling relations to find the form of the solution $q(t)$ in terms of $q(0)$, $\dot{q}(0)$ and ω_0.

(d) Consider the case where $\zeta = 0$ and $u(t) = \sin\omega t$, $\omega > \omega_0$. Solve for $z_1(\tau)$, the normalized output of the oscillator, with initial conditions $z_1(0) = z_2(0) = 0$ and use this result to find the solution for $q(t)$.

(Parts (a) and (b) are from AM 2.6.)