CDS 101/110a: Lecture 8-2
Loop Shaping/Frequency Design Methods

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Goals:

- Introduce loop shaping design and performance measures
- Short review PID (Proportional + Integral + Derivative) control
- Show how to design using loop shaping methodology

Reading:

- Åström and Murray, *Feedback Systems,* Ch 11
- *Advanced*: Lewis, Chapter 12

Homework: ?

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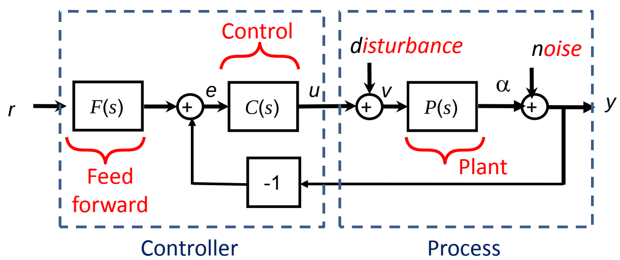
General loop transfer functions



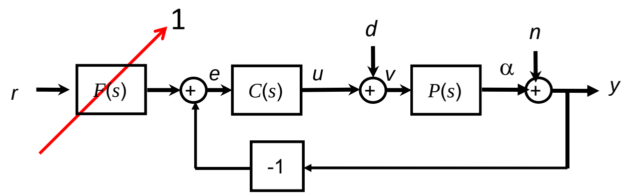
r = reference input
e = error
u = control
v = control + disturbance
α = true output
y = measured output

$$\begin{pmatrix} y \\ \alpha \\ v \\ u \\ e \end{pmatrix} = \begin{pmatrix} \frac{PCF}{1+PC} & \frac{P}{1+PC} & \frac{1}{1+PC} \\ \frac{PCF}{1+PC} & \frac{P}{1+PC} & \frac{-PC}{1+PC} \\ \frac{CF}{1+PC} & \frac{1}{1+PC} & \frac{-C}{1+PC} \\ \frac{CF}{1+PC} & \frac{-PC}{1+PC} & \frac{-C}{1+PC} \\ \frac{F}{1+PC} & \frac{-P}{1+PC} & \frac{-1}{1+PC} \end{pmatrix} \begin{pmatrix} r \\ d \\ n \end{pmatrix}$$

"Gang of Six"

Response of (y, u) to r	Response of u to (d,n)	Response of y to (d,n)
$\mathrm{TF} = \frac{PCF}{1+PC}$	$\mathrm{T} = \frac{PC}{1+PC}$	$\mathrm{PS} = \frac{P}{1+PC}$
$\mathrm{CFS} = \frac{CF}{1+PC}$	$\mathrm{CS} = \frac{C}{1+PC}$	$\mathrm{S} = \frac{1}{1+PC}$

Key loop transfer functions



Sensitivity: Function	$H_{er} = \text{S(s)} = \frac{1}{1+L(s)}$	
Complementary Sensitivity Function:	$H_{yr} = \text{T(s)} = \frac{L(s)}{1+L(s)}$	
Load Sensitivity Function:	$H_{yd} = \text{PS(s)} = \frac{P(s)}{1+L(s)}$	
Noise Sensitivity Function:	$H_{yn} = \text{CS (s)} = \frac{C(s)}{1+L(s)}$	

$$L(s) = P(s)C(s)$$

"Gang of Four"

Characterize most performance criteria of interest

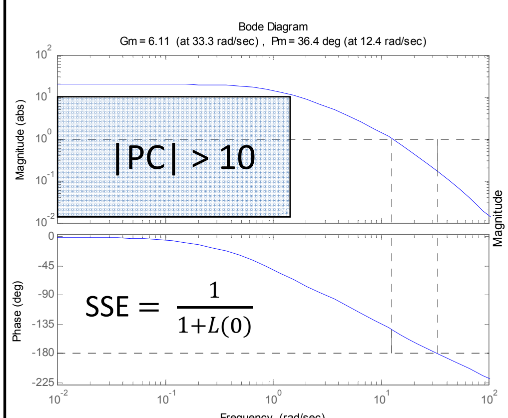
Example:

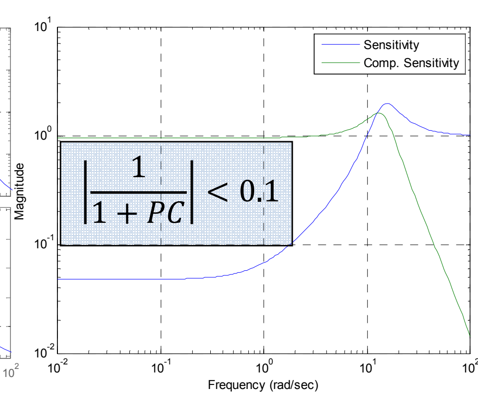
Suppose

$$L(s) = \frac{20}{(s+1)(s/10+1)(s/100+1)}$$

$$\text{S(s)} = \frac{1}{1+L(s)}$$

$$\text{T(s)} = \frac{L(s)}{1+L(s)}$$





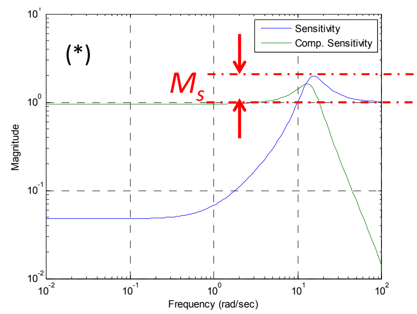
Sensitivity

Note: $$H_{yd} = \frac{P(s)}{1+L(s)} = P(s)\frac{1}{1+L(s)} = P(s)S(s)$$

- I.e., Closed Loop Response to Disturbance = Open Loop Response x Sensitivity
- I.e., S(s) tells us how variations in output are influenced by feedback
  - Disturbances with |S(iω)|<1 attenuated by feedback
  - Disturbances with |S(iω)|>1 amplified

- Max Sensitivity = max |S(iω)| := M_s
  - $M_s = 1/s_m$,
  - s_m is *stability margin*

$$L(s) = \frac{20}{(s+1)(s/10+1)(s/100+1)} \quad (*)$$



(*) Rowley & Battin, *Fundamentals & Applications of Modern Flow Control*, Ch 5

Sensitivity

Example plotted is: $L(s) = \frac{20}{(s+1)(s/10+1)(s/100+1)}$ (*)

$$S(s) = \frac{1}{1+L(s)}$$

- |S(iw)|>1 Inside unit circle centered at -1
- |S(iw)|<1 outside unit circle centered at -1

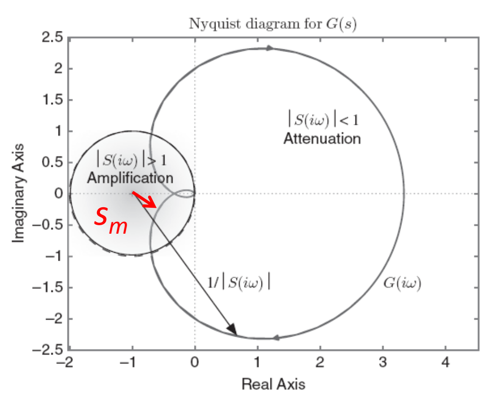
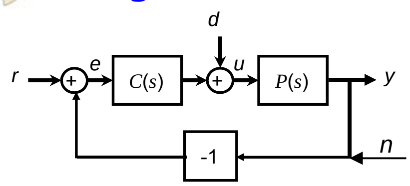


Fig. 4 Nyquist plot of the loop gain $G(s) = P(s)C(s)$ for the system (29). For frequencies for which $G(i\omega)$ enters the unit circle centered about the −1 point, disturbances are amplified and, for frequencies for which $G(s)$ lies outside this circle, disturbances are attenuated relative to open-loop.

(*) From Rowley & Battin, *Fundamentals & Applications of Modern Flow Control*, Ch 5

Design based on loop transfer function



$$L(s) = P(s)C(s)$$

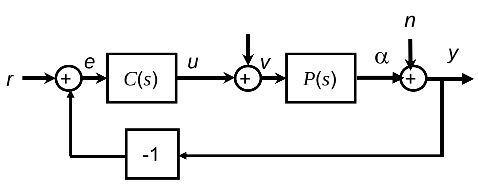
$$H_{er} = \frac{1}{1+L} \qquad H_{yr} = \frac{L}{1+L}$$

$$H_{yd} = \frac{P}{1+L} \qquad H_{yn} = \frac{-L}{1+L}$$

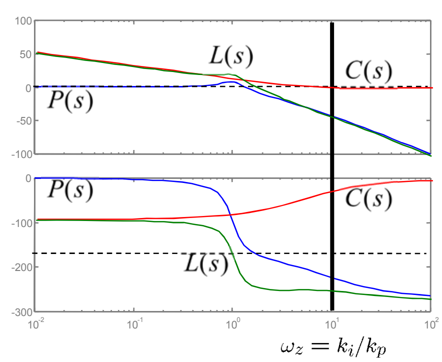
Gang of Four

- **Stability** depends only on *L* (s)= *P*(s)*C*(s) (last week)
  - Nyquist Diagram analyzes stability, and rough gain/phase margin performance
  - When gain or phase margin small, usually get large overshoot or ringing
- **Performance** depends (mostly) on *L = PC*
  - When L is large, tracking performance and disturbance rejection is good
  - When L is small, sensor noise rejection is good, actuator response is small.
  - Typically care about tracking and disturbance response at low frequencies, and sensor noise at high frequencies.

Design based on loop transfer function

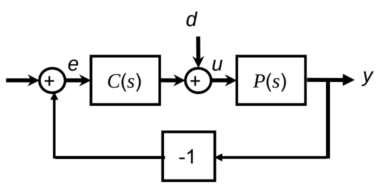


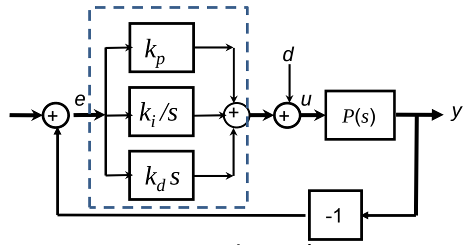
Since stability is determined by L(s) = P(s) C(s), and performance dominated by L(S), "design" C(s) so that L(s) has wanted spec.s



Proportional Integral Derivative (PID)

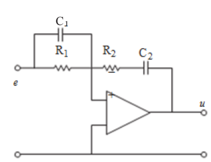
$$u(t) = \underbrace{\overbrace{k_p e(t)}^{\text{Current Error}}}_{\text{Proportional}} + \underbrace{\overbrace{k_i \int_0^t e(\tau)\, d\tau}^{\text{Steady State Error}}}_{\text{Integral}} + \underbrace{\overbrace{k_d \frac{de(t)}{dt}}^{\text{Anticipate/Damp}}}_{\text{Derivative}}$$



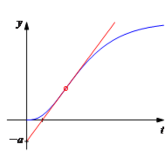


- Extremely common (includes PI or PD as special case)
- Can arbitrarily set poles of 1st and 2nd order systems
- Many tools for tuning PID loops and designing gains
- Found in biological Systems

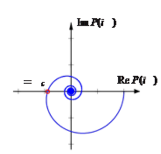
AutoTuning (PID)



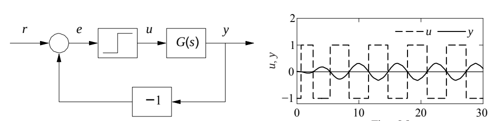
8-Axis Galil Motion Control "Board"



(a) Step response method



(b) Frequency response method



(a) Relay feedback (b) Oscillatory response

See Astrom & Murray pp. 302-306

Typical Performance Specifications

- Steady State Error
  - Sets zero frequency (or "DC") gain
  - "Less than X% error" => $1 + L(0) > 100/X$
- Tracking Error (up to some frequency)
  - "Less than Y% up to frequency f_T" => $L(i\omega) > 100/Y$ for $\omega < 2\pi f_T$
- Closed Loop Bandwidth, ω_b
  - Specify in different ways. For low pass: $|T(\omega)| > \frac{1}{\sqrt{2}}|T(0)|$ for $0 < \omega < \omega_b$
- Rise time, Overshoot, Settling Time (on step input)
  - Usually confirmed after design. Complex relation with freq. response.
- Stability Margins (*Robustness)*
  - Gain Margin of Factor Z
  - Phase Margin of U degrees (typically at least 30^o – 60^o)

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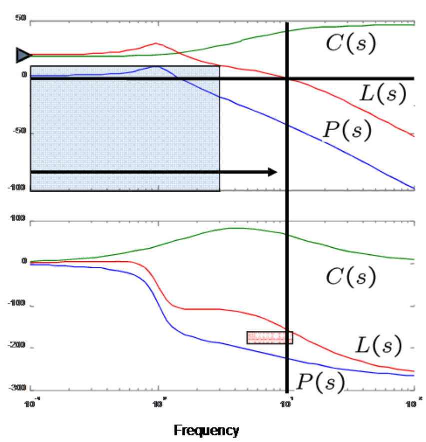
Graphical Overview of Loop Shaping



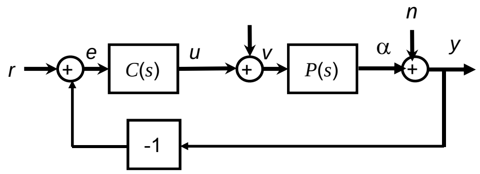
Performance Spec.s

- ▶ Steady state error
- Tracking Error
- ➜ Bandwidth
- Relative Stability

Approach

- P(s) + spec.s are given
- L(s) = P(s) C(s)
  - Craft C(s) to yield shape of L(s) which meets specs.
  - There are restrictions!

Algebraic Constraints on Performance

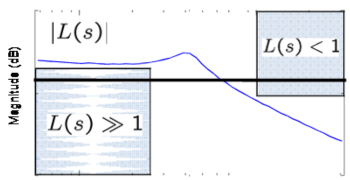


$$S(s) = H_{er} = \frac{1}{1+L(s)}$$

$$T(s) = H_{yr} = H_{an} = \frac{L(s)}{1+L(s)}$$

Goal: keep S(s) & T(s) "small"

- S(s) small => low tracking error
- T(s) small => good noise rejection



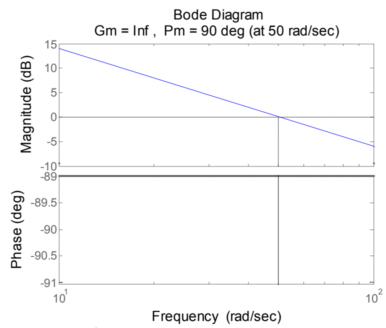
Problem: S(s) + T(s) = 1 !

- Can't make both S,T small at same freq.
- *Solution:* Keep S(s) small at low freq. and T(s) small at high freq.
- *Loop gain interpretation:* Make L(s) large at low freq. and small at high freq.

Process Inversion

- **Trick:** *Invert the Process/Plant*
  - Write Performance Spec.s as desired loop transfer, $L_D(s)$
  - Choose controller C(s) by *inverting* the plant:

    $C(s) = L_D(s)/P(s)$



- **Pros:**
  - Simple.
  - L(s) = k/s often suffices
  - Can use as 1^{st} cut design, followed by more tuning
- **Cons:**
  - Controllers usually high order (at least same order as plant)
  - Requires *perfect* model of plant/process
  - ***Does not work*** with RHP poles or zeros (design not *internally stable*)

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Better Loop Shaping Design Process

A Process: sequence of (nonunique) steps

1. Start with plant and performance specifications
2. If plant not stable, first stabilize it (e.g., PID)
3. Adjust/increase simple gains
   - Increase proportional gain for tracking error
   - Introduce integral term for steady-state error
   - Will derivative term improve overshoot?
4. Analyze/adjust for stability and/or phase margin
   - Adjust gains for margin
   - Introduce *Lead* or *Lag Compensators* to adjust phase margin at crossover and other critical frequencies
   - Consider PID if you haven't already

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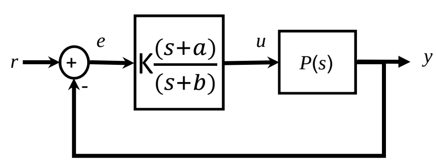
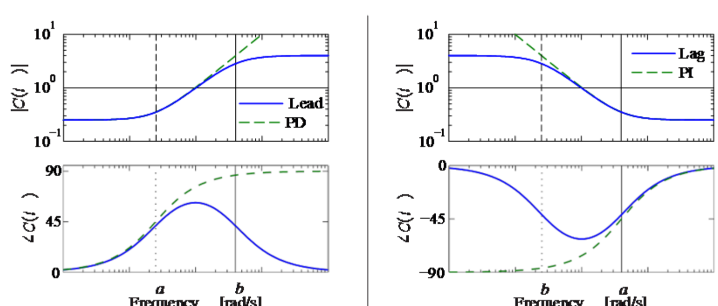
Lead & Lag Compensators

Lead: K > 0, a < b
- Add phase near crossover

Lag: K > 0, a> b
- Add gain in low frequencies



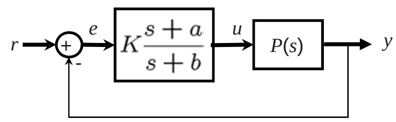


(a) Lead compensation, $a < b$

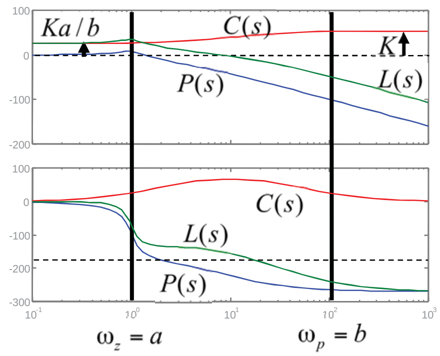
(b) Lag compensation, $b < a$

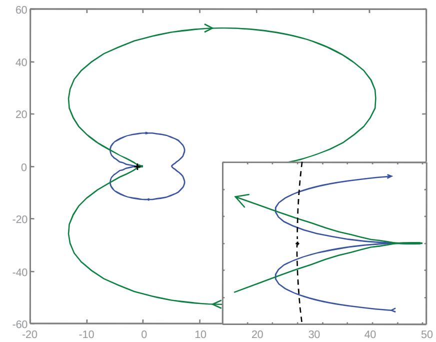
Lead compensation

- Use to increase phase in frequency band
  - Effect: lifts phase by increasing gain at high frequency
  - *Very* useful controller; increases PM
  - Bode: add phase between zero and pole
  - Nyquist: increase phase margin



$a < b \qquad K > 0$





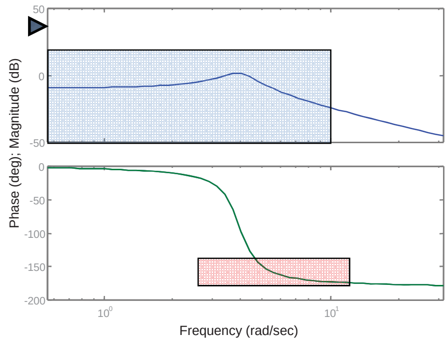
Example: Control of Vectored Thrust Aircraft

- System description
  - Vector thrust engine attached to wing
  - Inputs: fan thrust, thrust angle (vectored)
  - Outputs: position and orientation
  - States: *x*, *y*, θ + derivatives
  - Dynamics: flight aerodynamics

Control approach

- Design “inner loop” control law to regulate pitch (θ) using thrust vectoring
- Second “outer loop” controller regulates the position and altitude by commanding the pitch and thrust
- Basically the same approach as aircraft control laws

Performance Specification and Design Approach



$$P(s) = \frac{r}{Js^2 + ds + mg\ell}$$

Performance Specification

- $\leq$ 1% steady state error
  - Zero frequency gain > 100
- $\leq$ 10% tracking error up to 10 rad/sec
  - Gain > 10 from 0-10 rad/sec
- $\geq$ 45˚ phase margin
  - Gives good relative stability
  - Provides robustness to uncertainty

- Design approach
  - If choose C(s)=K, then poor phase margin
  - Add phase lead in 5-50 rad/sec range
  - Increase the gain to achieve steady state and tracking performance specs

$$C(s) = K\frac{s+a}{s+b}$$

$a = 25$

$b = 300$

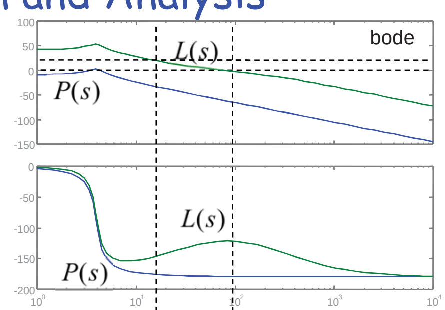
$K = 15 \times 300$

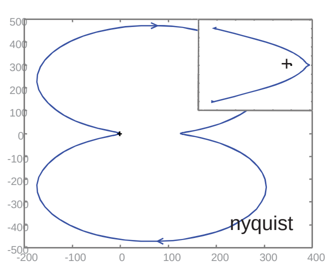
Control Design and Analysis

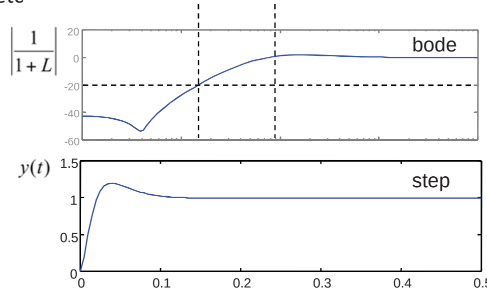
- Select parameters to satisfy specs
  - Place phase lead in desired crossover region (given by desired BW)
  - Phase lead peaks at $\omega = \sqrt{ab}$
  - Maximum phase depends on pole/zero ratio:

$$\phi_{\max} = 90^\circ - 2\tan^{-1}\sqrt{a/b}$$

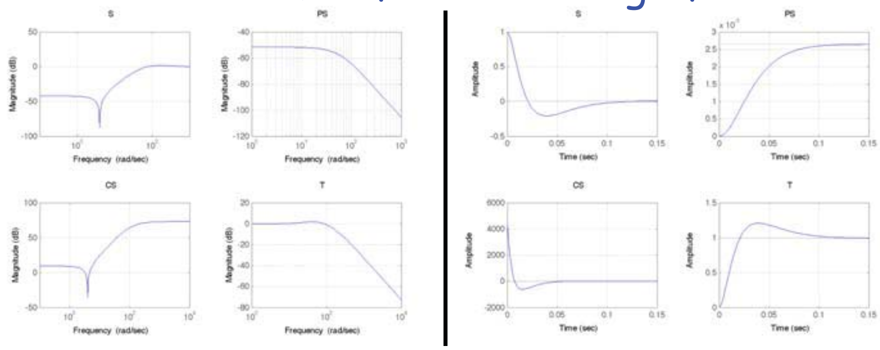
  - Set gain as needed for tracking + BW
  - Verify controller using Nyquist plot, etc







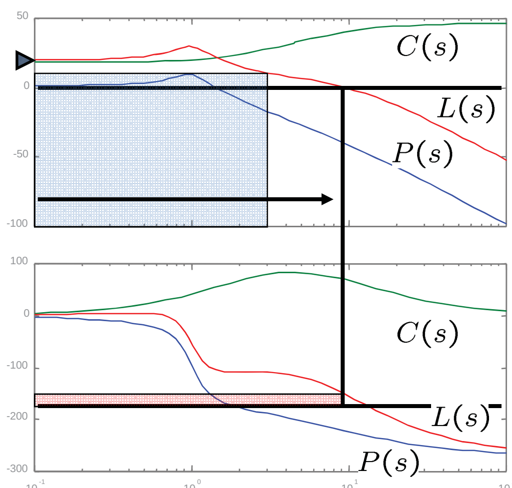
Control Verification: Gang of 4



- Remarks
  - Check each transfer function to look for peaks, large magnitude, etc
  - Example: Noise sensitivity function (CS) has very high gain; step response verifies poor step response
  - Implication: controller amplifies noise at high frequency $\Rightarrow$ will generate *lots* of motion of control actuators (flaps)
  - Fix: roll off the loop transfer function faster (high frequency pole)

Summary: Loop Shaping

- Loop Shaping for Stability & Performance
  - Steady state error, bandwidth, tracking



Main ideas

- Performance specs give bounds on loop transfer function
- Use controller to shape response
- Gain/phase relationships constrain design approach
- Lead compensator useful to add phase