



CDS 101/110a: Lecture 4-2 Observability and State Estimation

Douglas G. MacMartin

- Goals:
 - Define observability and give conditions for checking observability for linear control systems
 - Introduce the state estimation (aka observer) problem
 - Provide examples of state estimation in the context of closed loop design
- Reading:
 - Åström and Murray, Feedback Systems, Sections 7.1-7.3

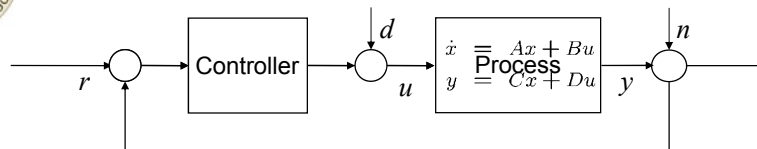
10/23/2013

D. MacMartin cds 101/110a 2013

1



First Questions...



- 1) Can the input u affect the dynamics?

$$\begin{aligned} \dot{x}_1 &= x_1 + u \\ \text{e.g. } \dot{x}_2 &= x_2 \quad \Rightarrow \text{Can't change } x_2 \end{aligned}$$

Equivalent to asking whether there is a u that allows us to reach any point in the state-space

- \Rightarrow Reachability (Monday), depends on A, B
- \Rightarrow Related to the design of state feedback $u = -Kx$

- 2) Does the measurement y contain enough information about the system?


$$\begin{aligned} \dot{x}_1 &= x_1 \quad y = x_1 \\ \text{e.g. } \dot{x}_2 &= x_2 \quad \Rightarrow \text{Can't measure } x_2 \end{aligned}$$

- \Rightarrow Observability (today), depends on A, C
- \Rightarrow Related to the design of observers to estimate state from measurement

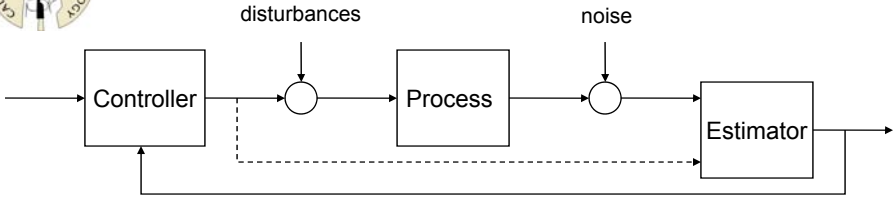
10/20/2013

D. MacMartin, CDS 101/110a 2013

2



The State Estimation Problem



- Problem Setup
 - Given a dynamical system with noise and uncertainty, estimate the state


$$\dot{x} = Ax + Bu + Fv \quad \longrightarrow \quad \hat{x} = \alpha(\hat{x}, y, u) \quad \leftarrow \text{estimator}$$

$$y = Cx + Du + Gw \quad \longrightarrow \quad \lim_{t \rightarrow \infty} E(x - \hat{x}) = 0$$
 - \hat{x} is called the estimate of x
 - $\leftarrow \text{expected value}$
- Remarks
 - Several sources of uncertainty: noise, disturbances, process, initial condition
 - 110a: Seek estimator that converges in absence of noise: $\lim_{t \rightarrow \infty} \hat{x} = x$
 - Uncertainties are unknown, except through their effect on measured output
 - First question: when is this even possible?

10/23/2013

D. MacMartin cds 101/110a 2013

3



Observability

- Definition: A dynamical system of the form:

$$\begin{aligned} \dot{x} &= f(x, u) \\ y &= h(x, u) \end{aligned}$$
- is observable if for any $T > 0$ it is possible to determine the state of the system $x(T)$ through measurements of $y(t)$ and $u(t)$ on the interval $[0, T]$
- Remarks
 - Observability must respect causality: only get to look at past measurements
 - Each initial condition must generate a unique output y
 - Start with ignoring noise, disturbances \Rightarrow estimate exact state
- Test to check observability for linear system:

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx \end{aligned}$$
- Theorem: A linear system is observable iff the observability matrix W_o is full rank, where:

$$W_o = \begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{n-1} \end{bmatrix}$$

10/23/2013

D. MacMartin cds 101/110a 2013

4



Proof of Observability Rank Condition, (1 of 2)

- Theorem A linear system is observable if and only if the observability matrix W_o is full rank.

- Proof (sufficiency) Write the output in terms of the convolution integral

$$y(t) = Ce^{At}x(0) + \int_0^t Ce^{A(t-\tau)}Bu(\tau)d\tau + Du(t).$$

- Since we know $u(t)$, we can subtract off its contribution and write

$$\tilde{y}(t) = Ce^{At}x(0)$$

- Now differentiate the (new) output and evaluate at $t = 0$

$$\tilde{y}(0) = Cx(0)$$

$$\dot{\tilde{y}}(0) = CAx(0)$$

$$\vdots$$

$$\tilde{y}^{(n)}(0) = CA^{n-1}x(0)$$

- Finally, invert to solve for $x(0)$. To find $x(T)$, use $x(T) = e^{AT}x(0)$.

10/23/2013

D. MacMartin cds 101/110a 2013

5



Proof of Observability Rank Condition, (2 of 2)

- Theorem A linear system is observable if and only if the observability matrix W_o is full rank.

- Proof (necessity) Again, we start with the convolution integral

$$y(t) = Ce^{At}x(0) + \int_0^t Ce^{A(t-\tau)}Bu(\tau)d\tau + Du(t).$$

- Subtracting off the input as before and expanding the exponential, we have

$$\tilde{y}(t) = Ce^{At}x(0) = C(I + At + \frac{1}{2}A^2t^2 + \dots + \frac{1}{k!}A^k t^k + \dots)x(0)$$

- By the Cayley-Hamilton theorem, we can write A^n in terms of lower powers of A and so we can write


$$\tilde{y}(t) = (\alpha_0(t)C + \alpha_1(t)CA + \dots + \alpha_{n-1}(t)CA^{n-1})x(0)$$

- If W_o is not full rank, then can choose $x(0) \neq 0$ such that $\tilde{y}(t) = 0 \Rightarrow$ not observable (since $x(0) = 0$ would produce the same output).

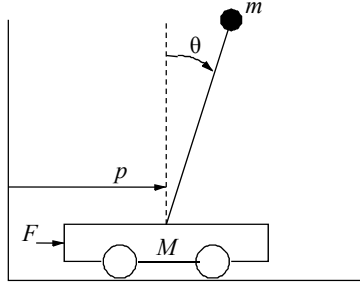
10/23/2013

D. MacMartin cds 101/110a 2013

6



Example #1: Linearized pendulum on a cart



- Question: can we determine the state of the system $(p, \dot{p}, \theta, \dot{\theta})$ from measuring position p ? How about from θ ?
- Approach: look at the linearization around the upright position (good approximation to the full dynamics if θ remains small)


$$\frac{d}{dt} \begin{bmatrix} p \\ \theta \\ \dot{p} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & \frac{m^2 \ell^2 g}{M_t J_t - m^2 \ell^2} & \frac{-c J_t}{M_t J_t - m^2 \ell^2} & \frac{-\gamma J_t \ell m}{M_t J_t - m^2 \ell^2} \\ 0 & \frac{M_t m g \ell}{M_t J_t - m^2 \ell^2} & \frac{-c \ell m}{M_t J_t - m^2 \ell^2} & \frac{-\gamma M_t}{M_t J_t - m^2 \ell^2} \end{bmatrix} \begin{bmatrix} p \\ \theta \\ \dot{p} \\ \dot{\theta} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \frac{J_t}{M_t J_t - m^2 \ell^2} \\ \frac{\ell m}{M_t J_t - m^2 \ell^2} \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix} x$$

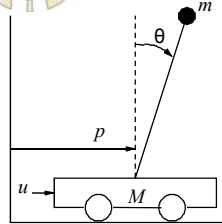
10/23/2013

D. MacMartin cds 101/110a 2013

7



Example #1, con't: Linearized pendulum on a cart



$$\frac{d}{dt} \begin{bmatrix} p \\ \theta \\ \dot{p} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & \frac{m^2 \ell^2 g}{\mu} & \frac{-c J_t}{\mu} & \frac{-\gamma J_t \ell m}{\mu} \\ 0 & \frac{M_t m g \ell}{\mu} & \frac{-c \ell m}{\mu} & \frac{-\gamma M_t}{\mu} \end{bmatrix} \begin{bmatrix} p \\ \theta \\ \dot{p} \\ \dot{\theta} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \frac{J_t}{\mu} \\ \frac{\ell m}{\mu} \end{bmatrix} u$$

$\mu = M_t J_t - m^2 \ell^2$

• Simplify by setting $c, \gamma = 0$


- Observability matrix for $C = [1 \ 0 \ 0 \ 0]$:

$$W_o = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & \frac{m^2 \ell^2 g}{\mu} & 0 & 0 \\ 0 & 0 & 0 & \frac{m^2 \ell^2 g}{\mu} \end{bmatrix} \begin{matrix} C \\ CA \\ CA^2 \\ CA^3 \end{matrix}$$
- Full rank \Rightarrow observable
- How about with $C = [0 \ 1 \ 0 \ 0]$?

$$W_o = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & \frac{M_t m g \ell}{\mu} & 0 & 0 \\ 0 & 0 & 0 & \frac{M_t m g \ell}{\mu^2} \end{bmatrix}$$

10/23/2013

D. MacMartin cds 101/110a 2013



State Estimation: Full Order Observer


- Given that a system is observable, how do we actually estimate the state?
 - Key insight: if current estimate is correct, follow the dynamics of the system

$$\dot{x} = Ax + Bu \quad \hat{\dot{x}} = \underbrace{A\hat{x} + Bu}_{\text{prediction (copy of dynamics)}} + L(y - C\hat{x}) \leftarrow \text{correction (based on output error)}$$

$$y = Cx$$
 - Modify the dynamics to correct for error based on a linear feedback term
 - L is the observer gain matrix; determines how to adjust the state due to error
 - Look at the error dynamics for $\tilde{x} = x - \hat{x}$ to determine how to choose L:

$$\dot{\tilde{x}} = \dot{x} - \dot{\hat{x}} = Ax + Bu - (A\hat{x} + Bu + LC(x - \hat{x})) = (A - LC)\tilde{x}$$
- Thm: If the pair (A, C) is observable (associated W_o is full rank), then we can place the eigenvalues of $A - LC$ arbitrarily through appropriate choice of L.
- Proof Note that the transpose of $A - LC$ is $A^T - C^T L^T$ and in this form, this is the same as the eigenvalue placement problem for state space controllers.
- Remark: In MATLAB, use $L' = \text{place}(A', C', \text{eigs})$ to determine L


10/23/2013 D. MacMartin cds 101/110a 2013 9



Comparison with Reachability & State Fdbk

- The pair (A,C) is observable $\Leftrightarrow (A^T, C^T)$ is reachable
 - Proofs are similar
- Dynamics with state feedback K are $(A - BK)$
Observer dynamics with gain L are $(A - LC)$
- The poles of $(A - LC)$ can be chosen arbitrarily \Leftrightarrow The poles of $(A^T - C^T L^T)$ can be chosen arbitrarily
 - Use $\text{place}(A^T, C^T, \lambda)$
 - Convergence depends on $\text{Re}(\lambda)$
- If (A,B) reachable, can transform to reachable canonical form.
If (A,C) observable, can transform to observable canonical form.
(Neither observability nor reachability depend on a change of variables)

10/23/2013 D. MacMartin cds 101/110a 2013 10



Simple Example

Double-integrator: $\ddot{z} = u$

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= u \\ y &= x_1 \end{aligned}$$

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

- Measure position, estimate full state (position and velocity)
- Simple guess:

$$\hat{x}_1 = y, \hat{x}_2 = \dot{y}$$
- Design observer:

$$\dot{\hat{x}} = A\hat{x} - L(y - C\hat{x}) + Bu$$

$$L = \text{place}(A', C', [-1; -1]) = \begin{bmatrix} 2 & 1 \end{bmatrix};$$


$$A - LC = \begin{bmatrix} -2 & 1 \\ 1 & 0 \end{bmatrix}$$

$$\begin{aligned} \dot{\hat{x}}_1 &= -2\hat{x}_1 + \hat{x}_2 + 2y \\ \dot{\hat{x}}_2 &= -\hat{x}_1 + y + u \end{aligned}$$

```

Wo=obsv(A,C);
L=place(A',C',[-1;-1.000001])';
sys=ss(A,B,C,0);
[u,t]=gensig('sin',5,20,dt);
[y,t,x]=lsim(sys,u,t);
y=y+0.1*randn(size(y));
% simple observer:
xh1=y;
xh2=diff([0;y])/dt;
% Luenberger observer:
Ao=A-L*C;
sys0=ss(Ao,[L B],eye(2),zeros(2,2));
xh=lsim(sys0,[y u],t);
            
```

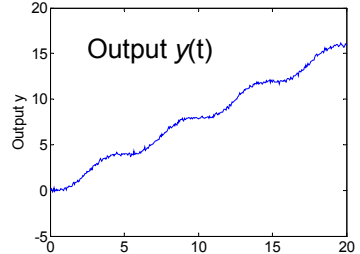
10/23/2013
D. MacMartin cds 101/110a 2013
11

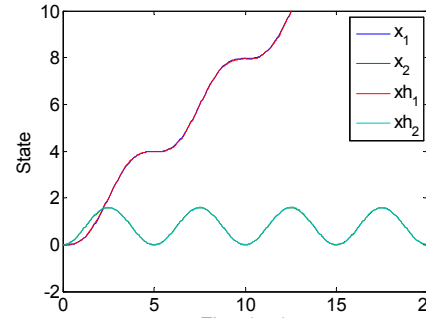


Guess which observer is which?

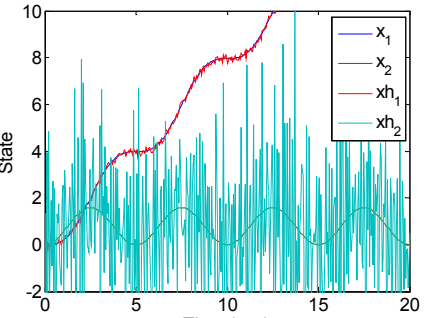
a) $\hat{x}_1 = y, \hat{x}_2 = \dot{y}$

b) $\dot{\hat{x}} = A\hat{x} - L(y - C\hat{x}) + Bu$






$\dot{\hat{x}} = A\hat{x} - L(y - C\hat{x}) + Bu$

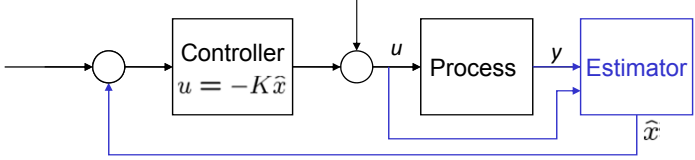


$\hat{x}_1 = y, \hat{x}_2 = \dot{y}$

10/23/2013
D. MacMartin cds 101/110a 2013
12




Separation Principle



- What happens when we apply state space controller using estimate of x ?
 - We assumed we measured x directly in analyzing controller; extra dynamics in the estimator could cause closed loop to go unstable
- Thm: If K is a stabilizing compensator for (A, B) and L gives a stable estimator for (A, C) , then the control law $u = -K\hat{x}$ is stable
 - This is an example of a separation principle: design the controller and estimator separately, then combine them and everything is OK
 - Be careful with signs on gains

10/23/2013
D. MacMartin cds 101/110a 2013
13



Proof of Separation Theorem


- Proof. Write down the dynamics for the complete system (assuming WLOG that $x_d, u_d = 0$):

$$\begin{aligned} \dot{x} &= Ax + Bu & \dot{\hat{x}} &= A\hat{x} + Bu + L(y - C\hat{x}) \\ y &= Cx & u &= -K\hat{x} + u_{\text{ref}} \end{aligned}$$
- Rewrite in terms of the error dynamics $\tilde{x} = x - \hat{x}$ and combined state x, \tilde{x} :

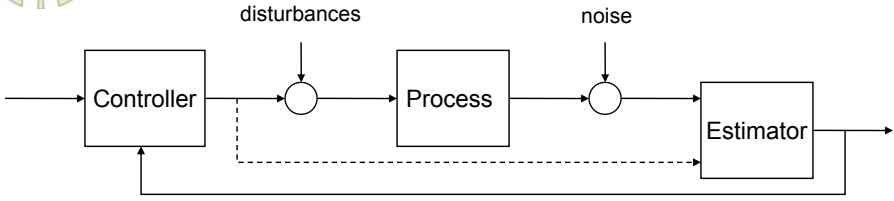
$$\dot{\tilde{x}} = (A - LC)\tilde{x} \quad \frac{d}{dt} \begin{bmatrix} x \\ \tilde{x} \end{bmatrix} = \begin{bmatrix} A - BK & BK \\ 0 & A - LC \end{bmatrix} \begin{bmatrix} x \\ \tilde{x} \end{bmatrix} + \begin{bmatrix} u_{\text{ref}} \\ 0 \end{bmatrix}$$
- Since the dynamics matrix is block diagonal, we find that the characteristic polynomial of the closed loop system is

$$\det(sI - A + BK) \det(sI - A + LC).$$
- This polynomial is a product of two terms, where the first is the characteristic polynomial of the closed loop system obtained with state feedback and the other is the characteristic polynomial of the observer error.
- Since each was designed to be stable \Rightarrow the entire system is stable
 - Note that stability does not guarantee good performance!

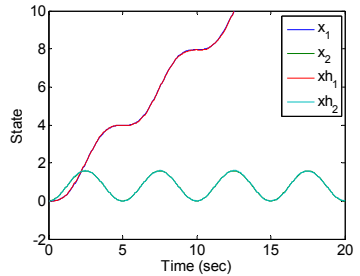
10/23/2013
D. MacMartin cds 101/110a 2013
14



Summary: Observers and State Estimation



- Observability
 - Derived conditions for when we could determine state from inputs & outputs:
check rank of observability matrix
- State Estimators
 - Construct state estimate based on prediction and correction (no noise yet)
- CDS110b: add noise to the problem
formulation \Rightarrow Kalman filter



10/23/2013
D. MacMartin cds 101/110a 2013
15