CDS 101/110a: Lecture 4-1
State Feedback

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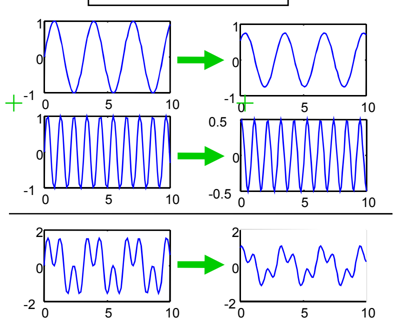
- Goals:
  - Define reachability of a control system
  - Give test(s) for reachability of linear systems and apply to examples
  - Describe the design of state feedback controllers for linear systems
- Reading:
  - Åström and Murray, Feedback Systems, Ch 6

Review from last week

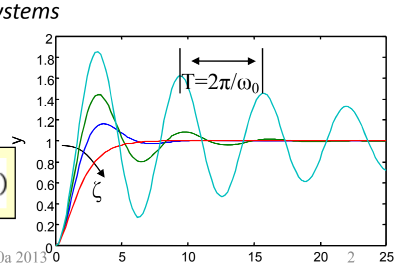
$$u \rightarrow \begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx + Du \\ x(0) &= 0 \end{aligned} \rightarrow y$$



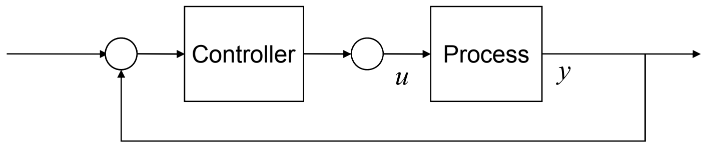
$$y(t) = Ce^{At}x(0) + \int_{\tau=0}^{t} Ce^{A(t-\tau)}Bu(\tau)\mathrm{d}\tau + Du(t)$$

Properties of Linear Systems:

- Linearity with respect to inputs and initial conditions
- Stability characterized by eigenvalues
- Response described by convolution integral
- Many applications and tools
- Characterizes a nonlinear system around an equilibrium point
  - *No system is linear for all input amplitudes, but linearization is a good enough model for many systems*

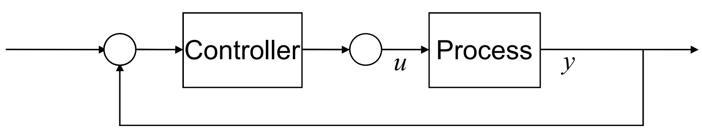


Control Overview



- Design controller so that:
  - i) System is stable
  - ii) Performance:
    - Keep close to equilibrium despite disturbances (disturbance rejection)
    - Move system to desired state (track reference)
  - iii) Robust to modeling errors
- Design framework:
  - Use time-domain (state-space) model (week 4-5, and cds110b)
  - Use frequency-domain information (weeks 5-10)

First Questions…



1) Can the input u affect the dynamics?

e.g.
$$\begin{aligned}\dot{x}_1 &= x_1 + u \\ \dot{x}_2 &= x_2\end{aligned}$$
$\Rightarrow$ Can't change x_2

Equivalent to asking whether there is a u that allows us to ***reach*** any point in the state-space

$\Rightarrow$ Reachability (today), depends on A, B

$\Rightarrow$ Related to the design of state feedback $u = -Kx$

2) Does the measurement y contain enough information about the system?

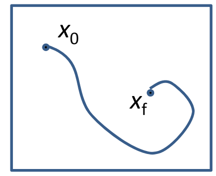
e.g.
$$\begin{aligned}\dot{x}_1 &= x_1 \qquad y = x_1 \\ \dot{x}_2 &= x_2\end{aligned}$$
$\Rightarrow$ Can't measure x_2

$\Rightarrow$ Observability (Wednesday), depends on A, C

$\Rightarrow$ Related to the design of observers to estimate state from measurement

Reachability of Input/Output Systems

$$\begin{aligned} \dot{x} &= f(x,u) \quad x \in \mathbb{R}^n,\ x(0) \text{ given} \\ y &= h(x,u) \quad u \in \mathbb{R},\ y \in \mathbb{R} \end{aligned}$$



- **Def'n**: An input/output system is *reachable* if for any $x_0, x_f \in \mathbb{R}^n$ and any time $T > 0$ there exists an input $u_{[0,T]} \in \mathbb{R}^n$ such that the solution of the dynamics starting from $x(0)=x_0$ and applying input $u(t)$ gives $x(T)=x_f$.

- Remarks
  - In the definition, x_0 and x_f do not have to be equilibrium points $\Rightarrow$ we don't necessarily stay at x_f after time T.
  - Reachability is defined in terms of states $\Rightarrow$ doesn't depend on output
  - For *linear systems,* can characterize reachability by looking at the general solution:

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx + Du \end{aligned} \qquad x(T) = e^{AT}x_0 + \int_{\tau=0}^{T} e^{A(T-\tau)}Bu(\tau)d\tau$$

Discrete-time Reachability

- Can we choose a sequence u[0], u[1], u[2],… to achieve any desired x[N]?

Given

$$x[k+1] = Ax[k] + Bu[k] \qquad x[0] = x_0$$

Then

$$\begin{aligned} x[1] &= Ax[0] + Bu[0] \\ x[2] &= A^2x[0] + ABu[0] + Bu[1] \\ x[3] &= A^3x[0] + A^2Bu[0] + ABu[1] + Bu[2] \\ &\vdots \end{aligned}$$

Or

$$x[3] = A^3x[0] + \begin{bmatrix} B & AB & A^2B \end{bmatrix} \begin{bmatrix} u[2] \\ u[1] \\ u[0] \end{bmatrix}$$

E.g., with $x \in R^3$ then $[B\ AB\ A^2B]$ is square. If invertible (full rank) then can solve for u[0],…

Tests for Reachability

$$\begin{aligned} \dot{x} &= Ax + Bu \quad x \in \mathbb{R}^n,\ x(0) \text{ given} \\ y &= Cx + Du \quad u \in \mathbb{R},\ y \in \mathbb{R} \end{aligned} \qquad x(T) = e^{AT}x_0 + \int_{\tau=0}^{T} e^{A(T-\tau)}Bu(\tau)\mathrm{d}\tau$$

- **Thm**: A linear system is reachable if and only if the $n \times n$ *reachability matrix*

$$\begin{bmatrix} B & AB & A^2B & \cdots & A^{n-1}B \end{bmatrix}$$

  is full rank.

- Remarks
  - *Very* simple test to apply. In MATLAB, use ctrb(A,B) and check rank w/ det()
  - If this test is satisfied, we say "the pair (A,B) is reachable"
  - Some insight into the proof can be seen by expanding the matrix exponential

$$\begin{aligned} e^{A(T-\tau)}B &= \left(I + A(T-\tau) + \frac{1}{2}A^2(T-\tau)^2 + \cdots + \frac{1}{(n-1)!}A^{n-1}(T-\tau)^{n-1} + \cdots\right)B \\ &= B + AB(T-\tau) + \frac{1}{2}A^2B(T-\tau)^2 + \cdots + \frac{1}{(n-1)!}A^{n-1}B(T-\tau)^{n-1} + \cdots \end{aligned}$$

  - Test does not give a measure of how much control effort is required
  - Other tests for reachability also exist

Cayley-Hamilton Theorem

- For $x \in \mathbb{R}^n$ (so $A \in \mathbb{R}^{n\times n}$)
- The rank of

$$\begin{bmatrix} B & AB & A^2B & \cdots & A^{n-1}B \end{bmatrix} \in \mathbb{R}^{n\times n}$$

  is the same as the rank of

$$\begin{bmatrix} B & AB & A^2B & \cdots & A^{n-1}B & A^nB \end{bmatrix} \in \mathbb{R}^{n\times(n+1)}$$

- Cayley-Hamilton theorem: for any $A \in \mathbb{R}^{n\times n}$
  - the characteristic polynomial is

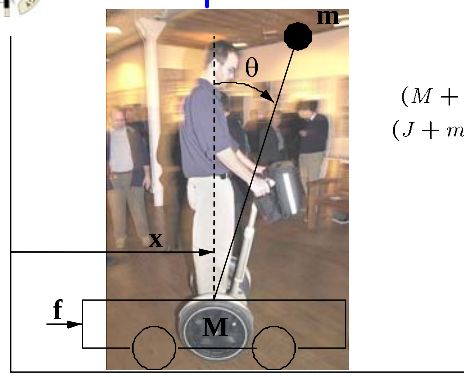
$$\det(sI - A) = s^n + a_1 s^{n-1} + \ldots + a_{n-1}s + a_n = 0$$

  - the matrix satisfies

$$A^n + a_1 A^{n-1} + \ldots + a_{n-1}A + a_n I = 0$$

  - which implies that for any $k \geq n$ then $A^k = \sum_{j=0}^{n-1} \alpha_j A^j$
- *If controllability matrix is rank-deficient, adding more terms doesn't help*

Example: Inverted Pendulum on a Cart

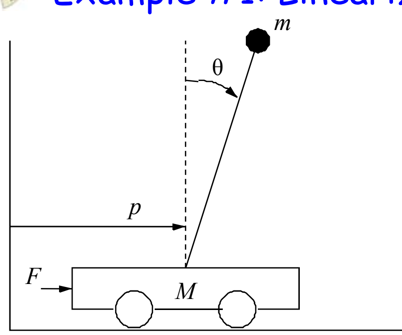


$$(M+m)\ddot{x} + ml\cos\theta\ddot{\theta} = -b\dot{x} + ml\sin\theta\dot{\theta}^2 + f$$
$$(J+ml^2)\ddot{\theta} + ml\cos\theta\ddot{x} = -mgl\sin\theta$$

- State: $x,\ \theta,\ \dot{x},\ \dot{\theta}$
- Input: *u* = *F*
- Output: *y* = *x*
- Linearize according to previous formula around θ = 0

$$\frac{d}{dt}\begin{bmatrix} x \\ \theta \\ \dot{x} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & \dfrac{m^2gl^2}{J(M+m)+Mml^2} & \dfrac{-(J+ml^2)b}{J(M+m)+Mml^2} & 0 \\ 0 & \dfrac{mgl(M+m)}{J(M+m)+Mml^2} & \dfrac{-mlb}{J(M+m)+Mml^2} & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ \dfrac{J+ml^2}{J(M+m)+Mml^2} \\ \dfrac{ml}{J(M+m)+Mml^2} \end{bmatrix} u$$
$$y = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix} x$$

Example #1: Linearized pendulum on a cart

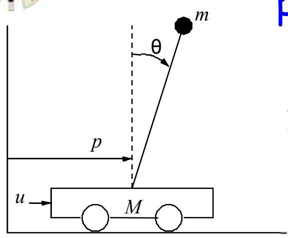


- Question: can we locally control the position of the cart by proper choice of input?
- Approach: look at the linearization around the upright position (good approximation to the full dynamics if θ remains small)

$$\frac{d}{dt}\begin{bmatrix} p \\ \theta \\ \dot{p} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & \frac{m^2l^2g}{M_tJ_t - m^2l^2} & \frac{-cJ_t}{M_tJ_t - m^2l^2} & \frac{-\gamma J_t lm}{M_tJ_t - m^2l^2} \\ 0 & \frac{M_t mgl}{M_tJ_t - m^2l^2} & \frac{-clm}{M_tJ_t - m^2l^2} & \frac{-\gamma M_t}{M_tJ_t - m^2l^2} \end{bmatrix} \begin{bmatrix} p \\ \theta \\ \dot{p} \\ \dot{\theta} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \frac{J_t}{M_tJ_t - m^2l^2} \\ \frac{lm}{M_tJ_t - m^2l^2} \end{bmatrix} u$$
$$y = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix} x,$$

Note: equations on previous slide did not include damping on angular rate

Example #1, con't: Linearized pendulum on a cart



$$\frac{d}{dt}\begin{bmatrix} p \\ \theta \\ \dot{p} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & \frac{m^2 l^2 g}{\mu} & \cancel{\frac{-cJ_t}{\mu}}^{\,0} & \cancel{\frac{-\gamma J_t lm}{\mu}}^{\,0} \\ 0 & \frac{M_t mgl}{\mu} & \cancel{\frac{-clm}{\mu}}^{\,0} & \cancel{\frac{-\gamma M_t}{\mu}}^{\,0} \end{bmatrix}\begin{bmatrix} p \\ \theta \\ \dot{p} \\ \dot{\theta} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \frac{J_t}{\mu} \\ \frac{lm}{\mu} \end{bmatrix} u$$

$$\mu = M_t J_t - m^2 l^2$$

- Simplify by setting c, γ = 0

- Reachability matrix

$$W_r = \begin{array}{c} \left[\begin{array}{c|c|c|c} 0 & \frac{J_t}{\mu} & 0 & \frac{gl^3m^3}{\mu^2} \\ 0 & \frac{lm}{\mu} & 0 & \frac{gl^2m^2(m+M)}{\mu^2} \\ \frac{J_t}{\mu} & 0 & \frac{gl^3m^3}{\mu^2} & 0 \\ \frac{lm}{\mu} & 0 & \frac{gl^2m^2(m+M)}{\mu^2} & 0 \end{array}\right] \\ \begin{array}{cccc} B & AB & A^2B & A^3B \end{array} \end{array}$$

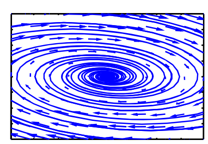
- Full rank as long as constants are such that columns 1 and 3 are not multiples of each other
- Reachable as long as $\det(W_r) = \dfrac{g^2\ell^4 m^4}{\mu^4} \neq 0$
- Can "steer" linearization between points by proper choice of input

Control Design Concepts

- System description: often single input, single output (MIMO also OK)

$$\begin{aligned} \dot{x} &= f(x,u) \quad && x \in \mathbb{R}^n,\ x(0) \text{ given} \\ y &= h(x,u) \quad && u \in \mathbb{R},\ y \in \mathbb{R} \end{aligned}$$

- Stability: stabilize the system around an equilibrium point
  - Given equilibrium point $x_e \in \mathbb{R}^n$, find control "law" $u = \alpha(x)$ such that $\lim_{t\to\infty} x(t) = x_e \ \forall x(0) \in \mathbb{R}^n$



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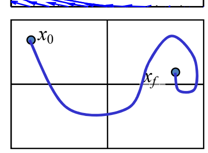
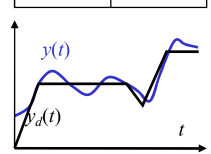
- Reachability: steer the system between two points
  - Given $x_0, x_f \in \mathbb{R}^n$, find an input $u(t)$ such that $\dot{x} = f(x, u(t))$ takes $x(t_0) = x_0 \rightarrow x(T) = x_f$



- Tracking: track a given output trajectory
  - Given $y_d(t)$, find $u = \alpha(x,t)$ such that $\lim_{t\to\infty} (y(t) - y_d(t)) = 0,\ \forall x(0) \in \mathbb{R}^n$



State space controller design for linear systems

$$\begin{aligned}\dot{x} &= Ax + Bu \quad x \in \mathbb{R}^n,\ x(0) \text{ given} \\ y &= Cx + Du \quad u \in \mathbb{R},\ y \in \mathbb{R}\end{aligned} \qquad x(T) = e^{AT}x_0 + \int_{\tau=0}^{T} e^{A(T-\tau)}Bu(\tau)\mathrm{d}\tau$$

- **Goal:** find a linear control law $u = -K\,x$ such that the closed loop system
  $$\dot{x} = Ax - BKx = (A - BK)x$$
  is stable at x_e=0.

- Remarks
  - Stability based on eigenvalues $\Rightarrow$ use K to make eigenvalues of $(A - BK)$ stable
  - Can also link eigenvalues to *performance* (e.g., initial condition response)
  - Question: when can we place the eigenvalues any place that we want?

- **Theorem:** The eigenvalues of $(A - BK)$ can be set to arbitrary values if and only if the pair (A, B) is reachable.

- MATLAB: K = place(A, B, eigs)

Reachable Canonical Form

- If the system is reachable, then there exists a transformation z = Tx such that:

$$\dot{z} = \begin{bmatrix} -a_1 & -a_2 & -a_3 & \cdots & -a_n \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0\cdots & 0 & \\ \vdots & & \ddots & \ddots & \vdots \\ 0 & & & 1 & 0 \end{bmatrix} z + \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} u$$

- (Check [B AB A^2B … A^{n-1}B]… is this system reachable?)
- Characteristic polynomial is: $\lambda(s) = s^n + a_1 s^{n-1} + \cdots + a_{n-1}s + a_n$

- Choose state feedback: $u = -Kz = -\begin{bmatrix} k_1 & k_2 & \cdots & k_n \end{bmatrix} z$
- Then closed-loop system is:

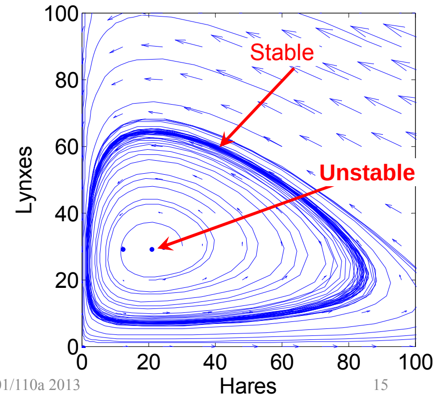
$$\dot{z} = \begin{bmatrix} -a_1 - k_1 & -a_2 - k_2 & -a_3 - k_3 & \cdots & -a_n - k_n \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0\cdots & 0 & \\ \vdots & & \ddots & \ddots & \vdots \\ 0 & & & 1 & 0 \end{bmatrix} z$$

Example #2: Predator prey

$$\frac{dH}{dt} = rH\left(1-\frac{H}{k_c}\right) - \frac{aHL}{c+H}, \qquad H \geq 0$$
$$\frac{dL}{dt} = b\frac{aHL}{c+H} - dL, \qquad L \geq 0$$

- Controlled dynamics: modulate food supply

$$\frac{dH}{dt} = (r+u)H\left(1-\frac{H}{k_c}\right) - \frac{aHL}{c+H}$$
$$\frac{dL}{dt} = b\frac{aHL}{c+H} - dL,$$



- Q1: can we move from some initial population of lynxes and hares to a specified one in time T by modulation of the food supply?
- Q2: can we stabilize the population around the desired equilibrium point
- Approach: try to answer this question locally, around the natural equilibrium point

Example #2: Problem setup

- Equilibrium point calculation

$$\frac{dH}{dt} = (r+u)H\left(1-\frac{H}{k_c}\right) - \frac{aHL}{c+H}$$
$$\frac{dL}{dt} = b\frac{aHL}{c+H} - dL,$$

  – x_e = (20.5,29.5), u_e = 0

```
% Compute the equil point
% predprey.m contains dynamics
f = inline('predprey(0,x)');
xeq = fsolve(f, [20, 30]);

% Compute linearization
A = [...];
B = [H0*(1 - H0/K); 0];
p = [-1;-2];
K = place(A,B,p)
```

- Linearization
  – Compute linearization around equil. point, x_e: $A = \left.\frac{\partial f}{\partial x}\right|_{x_e,u_e}$ $B = \left.\frac{\partial f}{\partial u}\right|_{x_e,u_e}$

$$\frac{d}{dt}\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} r - \frac{2H_0 r}{k} - \frac{aL_0}{c+H_0} + \frac{aL_0H_0}{(c+H_0)^2} & -\frac{aH_0}{c+H_0} \\ baL_0\left(\frac{1}{c+H_0} - \frac{H_0}{(c+H_0)^2}\right) & ba\frac{H_0}{c+H_0} - d \end{bmatrix}\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} + \begin{bmatrix} H_0\left(1-\frac{H_0}{k}\right) \\ 0 \end{bmatrix} v$$

  – Reachable? YES, if $ba \neq 0$ (check [B AB]) $\Rightarrow$ can locally steer to any point

Example #2: Stabilization via eigenvalue assignment

$$\frac{d}{dt}\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} r - \frac{2H_0 r}{k} - \frac{aL_0}{c+H_0} + \frac{aL_0H_0}{(c+H_0)^2} & -\frac{aH_0}{c+H_0} \\ baL_0\left(\frac{1}{c+H_0} - \frac{H_0}{(c+H_0)^2}\right) & ba\frac{H_0}{c+H_0} - d \end{bmatrix}\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} + \begin{bmatrix} H_0\left(1 - \frac{H_0}{k}\right) \\ 0 \end{bmatrix} v$$

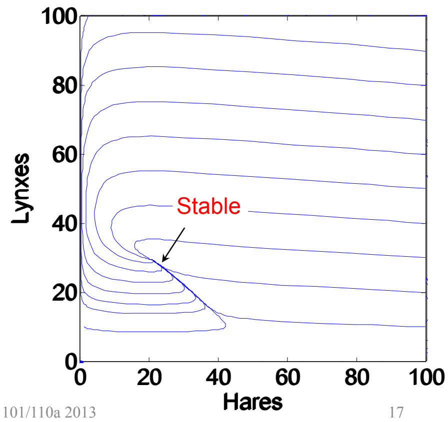
- Control design:

$$\begin{aligned} v &= -Kz + k_r r \\ u &= u_e - K(x - x_e) + k_r(r - y_e) \end{aligned}$$

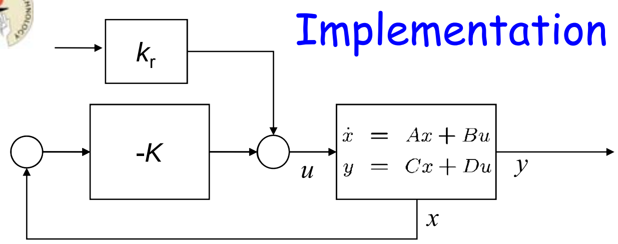
- Place poles at stable values
  - Choose λ = -1, -2
  - K = place(A, B, [-1; -2]);

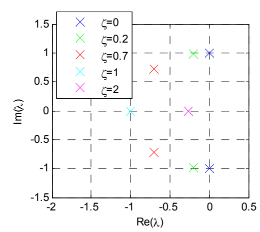
- Modify NL dynamics to include control

$$\begin{aligned} \frac{\mathrm{d}H}{\mathrm{d}t} &= (r+u)H\left(1 - \frac{H}{k_c}\right) - \frac{aHL}{c+H} \\ \frac{\mathrm{d}L}{\mathrm{d}t} &= b\frac{aHL}{c+H} - dL, \end{aligned}$$



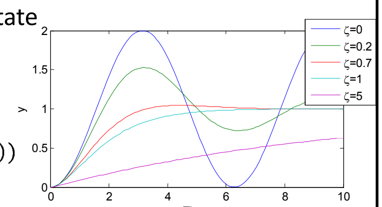
Implementation





Remarks:

1. In practice, don't always have access to full state
   → estimate state from measurement *y*
2. What to pick for eigenvalues?
   - For each eigenvalue $\lambda_i = \sigma_i + j\omega_i$, get contribution of the form $y_i(t) = e^{\sigma t}(a\sin(\omega t) + b\cos(\omega t))$
   - Faster response will require more control effort
   - Optimal control: LQR (in text, CDS 110b)



3. How to obtain desired tracking response so that $y_{ss} = r$ for some reference *r*?
   - Choose $u = -Kx + k_r r$
   - Steady state (if D=0): $y = Cx = C(A - BK)^{-1}Bk_r r \;\Rightarrow\; k_r = \mathrm{inv}[C(A - BK)^{-1}B]$