CDS 101/110a: Lecture 10-1
Limits on Performance

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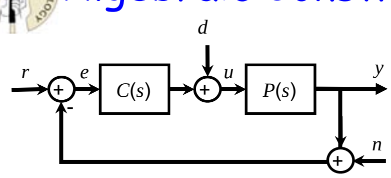
Goals:

- Describe limits of performance on feedback systems
- Introduce Bode's integral formula and the "waterbed" effect
- Show some of the limitations of feedback due to RHP poles and zeros

Reading:

- Åström and Murray, *Feedback Systems,* Ch 11

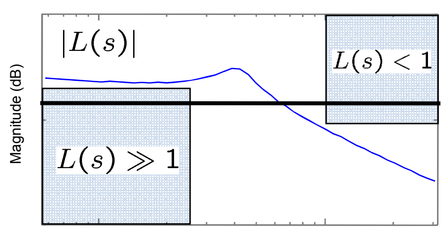
Algebraic Constraints on Performance



$$H_{er} = \frac{1}{1+PC} =: S$$ Sensitivity function

$$H_{yn} = \frac{PC}{1+PC} =: T$$ Complementary sensitivity function

- Goal: keep S & T small
  - S small ⇒ low tracking error
  - T small ⇒ good noise rejection (and robustness [CDS 110b])
- Problem: S + T = 1
  - Can't make *both* S & T small at the same frequency
  - Solution: keep S small at low frequency and T small at high frequency
  - Loop gain interpretation: keep L large at low frequency, small at high freq.



- Transition between large gain and small gain complicated by stability (phase margin)

Sensitivity

- From Rowley & Battin, *Fundamentals & Applications of Modern Flow Control*, Ch 5
- Example plotted is:

$$\frac{20}{(s+1)(s+2)(s+3)}$$

- Distance from -1 impacts performance as well as robustness

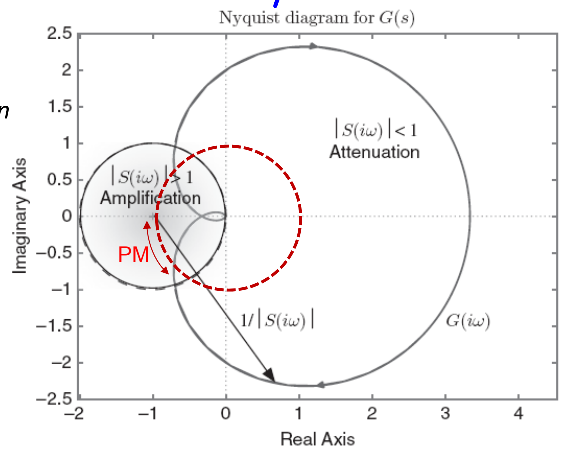
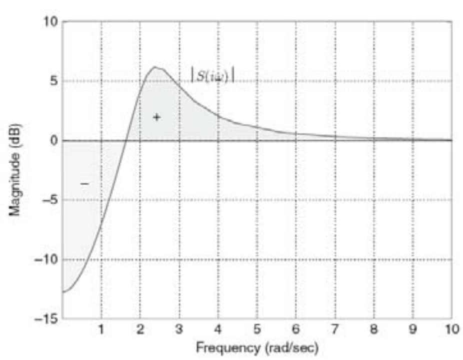


Fig. 4 Nyquist plot of the loop gain $G(s) = P(s)C(s)$ for the system (29). For frequencies for which $G(i\omega)$ enters the unit circle centered about the −1 point, disturbances are amplified and, for frequencies for which $G(s)$ lies outside this circle, disturbances are attenuated relative to open-loop.

Bode's Integral Formula: the Waterbed Effect

- Bode's integral formula for S = 1/(1+PC) = 1/(1+L):
  - Let p_k be the *unstable* poles of $L(s)$ and assume relative degree of $L(s) \geq 2$
  - Theorem: the area under the sensitivity function is a conserved quantity:

$$\int_0^\infty \log_e |S(j\omega)| d\omega = \int_0^\infty \log_e \frac{1}{|1+L(j\omega)|} d\omega = \pi \sum_{p_k \in \text{RHP}} \text{Re } p_k$$

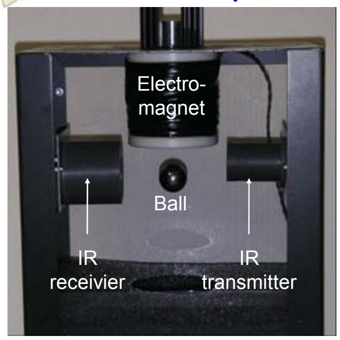


Waterbed effect:

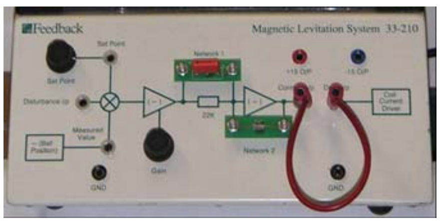
- Making sensitivity smaller over some frequency range requires *increase* in sensitivity someplace else
- Presence of RHP poles makes this effect worse
- Actuator bandwidth further limits what you can do
- Note: area formula is linear in ω; Bode plots are logarithmic

Fig. 5 Magnitude of $S(i\omega)$, illustrating the area rule (31): for this system, the area of attenuation (denoted −) must equal the area of amplification (denoted +), no matter what controller $C(s)$ is used.

Example: Magnetic Levitation

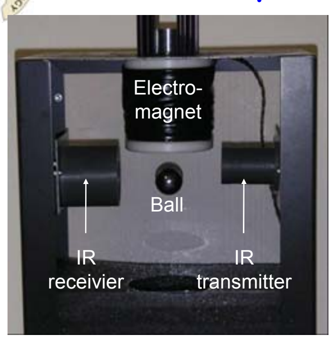


- System description
  - Ball levitated by electromagnet
  - Inputs: current thru electromagnet
  - Outputs: position of ball (from IR sensor)
  - States: $z, \dot{z}$
  - Dynamics: *F = ma, F* = magnetic force generated by wire coil



- Controller circuit
  - Active R/C filter network
  - Inputs: set point, disturbance, ball position
  - States: currents and voltages
  - Outputs: electromagnet current

Equations of Motion



- Process: actuation, sensing, dynamics

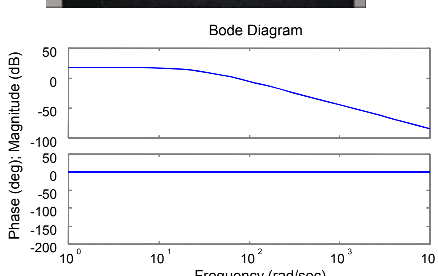
$$m\ddot{z} = mg - k_m(k_A u)^2/z^2$$

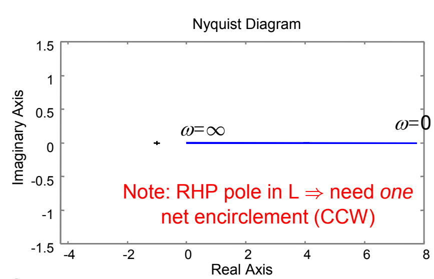
$$v_{ir} = k_T z + v_0$$

  - *u* = current to electromagnet
  - v_{ir} = voltage from IR sensor
- Linearization:

$$P(s) = \frac{-k}{s^2 - r^2} \qquad k, r > 0$$

  - Poles at $s = \pm r \Rightarrow$ open loop unstable

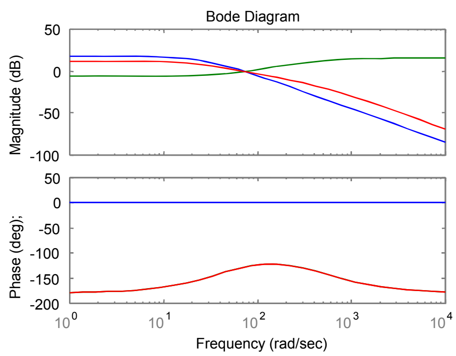


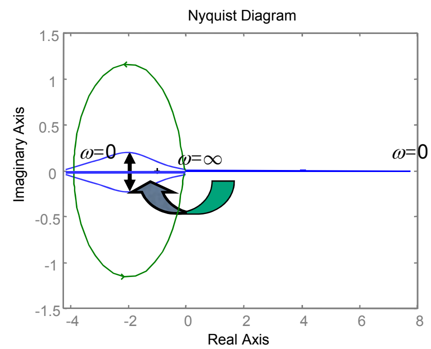


Control Design

- Need to *create* encirclement
  - Loop shaping is not useful here
  - Flip gain to bring Nyquist plot over -1 point
  - Insert phase to create CCW encirclement
- Can accomplish using a lead compensator
  - Produce phase lead at crossover
  - Generates loop in Nyquist plot

$$C(s) = -k\frac{s+a}{s+b}$$



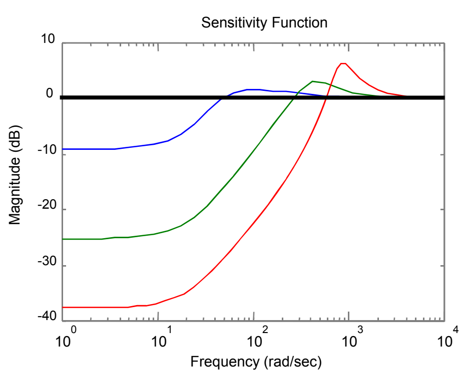


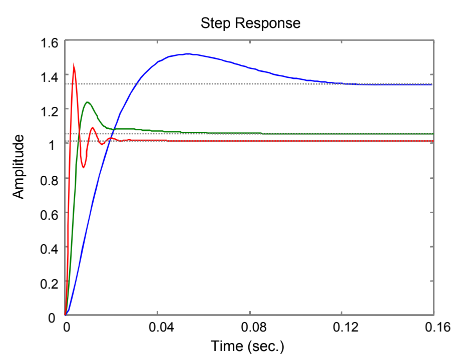
Performance Limits

- Nominal design gives low perf
  - Not enough gain at low frequency
  - Try to adjust overall gain to improve low frequency response
  - Works well at moderate gain, but notice waterbed effect
- Bode integral limits improvement

$$\int_0^\infty \log|S(j\omega)|d\omega = \pi r$$

  - Must increase sensitivity at some point

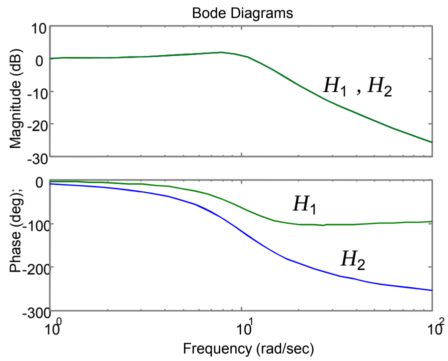


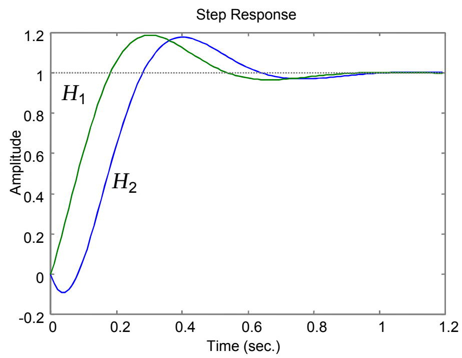


Right Half Plane Zeros

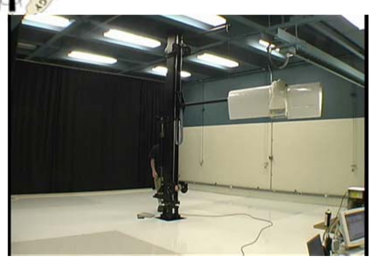
- Right half plane zeros produce “non-minimum phase” behaviour
  - Phase of frequency response has additional phase lag for given magnitude
  - Can cause output to move *opposite* from input for a short period of time
- Example: $H_1 = \dfrac{s+a}{s^2 + 2\zeta\omega_n s + \omega_n^2}$ vs $H_2 = \dfrac{-(s-a)}{s^2 + 2\zeta\omega_n s + \omega_n^2}$

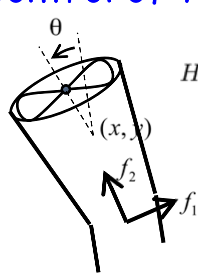
$$= H_1(s) \times \left(-\frac{s-a}{s+a}\right)$$





Example: Lateral Control of the Ducted Fan

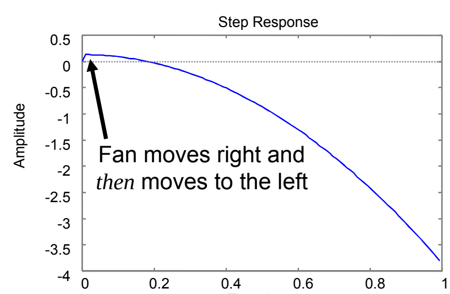




$$H_{xf_1} = \frac{s^2 - mg\ell}{Js^2 + ds - mg\ell}$$

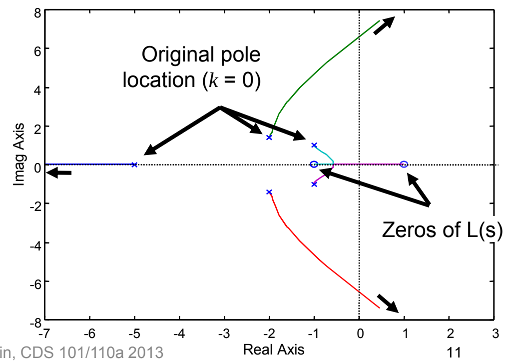
- Poles: 0, 0, -σ ± jω
- Zeros: $\pm\sqrt{mg\ell}$

- Source of non-minimum phase behavior
  - To move left, need to make θ > 0
  - To generate positive θ, need $f_1 > 0$
  - Positive f_1 causes fan to move *right* initially
  - Fan starts to move left after short time (as fan rotates)

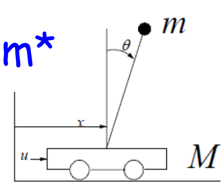


Stability in the Presence of Zeros

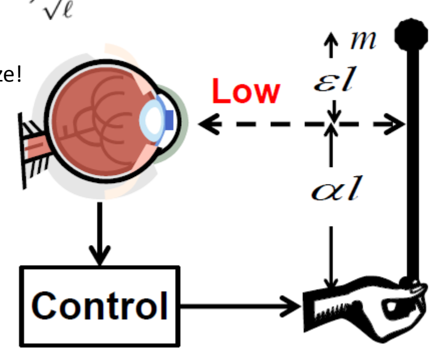
- Loop gain limitations
  - Poles of closed loop = zeroes of 1 + L. Suppose $C = k\, n_c/d_c$, where k is the gain of the controller
$$1 + L = 1 + k\frac{n_c n_p}{d_c d_p} = 1 + k\frac{n}{d} = \frac{d + kn}{d}$$
  - For large k, closed loop poles approach open loop zeros
  - RHP zeros limit maximum gain ⇒ serious design constraint!
- Root locus interpretation
  - Plot location of eigenvalues as a function of the loop gain k
  - Can show that closed loop poles go from open loop poles (k = 0) to open loop zeros (k = ∞)



Balancing an inverted pendulum*



- Looking at the end:
  - A longer pendulum is easier to stabilize; slower time constant… $p = \sqrt{g(M+m)/M}\frac{1}{\sqrt{\ell}}$
- Look at a fixed vertical height $\alpha\ell$
  - A longer pendulum is *harder* to stabilize!
- Dynamics:
$$\frac{\epsilon\ell s^2 - g}{s^2(M\ell s^2 - (M+m)g)}$$
  - RHP zero at $z = \sqrt{g/(\epsilon\ell)}$
- RHP pole *and* RHP zero is *very* hard
  - RHP pole gives a minimum bandwidth
  - RHP zero gives a maximum bandwidth



*Example from John Doyle

Further Constraints

- Because you can’t cancel a RHP pole with a zero, nor a RHP zero with a pole, then if the closed-loop system is stable we have:
  – If p is a RHP pole of L(s), then

    $$S(p)=0 \text{ and } T(p)=1$$

  – If z is a RHP zero of L(s), then

    $$S(z)=1 \text{ and } T(z)=0$$

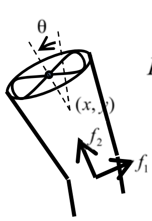
- This (roughly) implies a minimum bandwidth (>p) for any unstable pole
- And a maximum bandwidth (<z) for any non-minimum phase zero

Additional performance limits due to RHP zeros

- **Another waterbed-like effect:** look at maximum of H_{er} over frequency range:

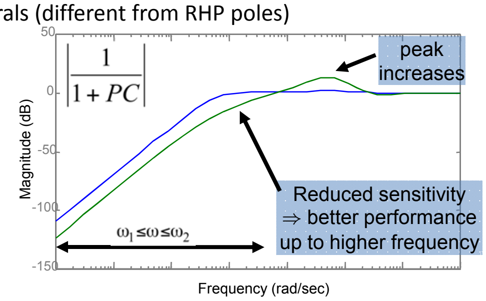
$$M_1 = \max_{\omega_1 \le \omega \le \omega_2} |H_{er}(j\omega)| \qquad M_2 = \max_{0 \le \omega \le \infty} |H_{er}(j\omega)|$$

- **Thm:** Suppose that P has a RHP zero at z. Then there exist constants c_1 and c_2 (depending on ω_1, ω_2, z) such that $c_1 \log M_1 + c_2 \log M_2 \ge 0$
– M_1 typically << 1 $\Rightarrow$ M_2 must be larger than 1 (since sum is positive)
– If we increase performance in active range (make M_1 and H_{er} smaller), we must lose performance (H_{er} increases) some place else
– Note that this affects *peaks* not integrals (different from RHP poles)



$$H(s) = \frac{(s^2 - mgl)}{s^2(Js^2 + ds + mgl)}$$

- Poles: 0, 0, $-\sigma \pm j\,\omega_d$
- Zeros: $\pm\sqrt{mgl}$



Summary: Limits of Performance

- Many limits to performance
  - Algebraic: S + T = 1
  - RHP poles: Bode integral formula
  - RHP zeros: Waterbed effect on peak of S

Main message: try to avoid RHP poles and zeros whenever possible (e.g., re-design)

$$\int_0^{\infty} \log_e |S(j\omega)| d\omega = \int_0^{\infty} \log_e \frac{1}{|1 + L(j\omega)|} d\omega = \pi \sum \text{Re } p_k$$