CDS 101 Midterm Review

November 2, 2012

1 Stability

We can determine stability of a linear system by looking at the eigenvalues. Please note that the statements below hold only under the assumption that there are no repeated eigenvalues unless otherwise stated.

asymptotically stable This holds if the $\mathbf{Re}(\lambda_i) < 0$ for all eigenvalues. This holds even when there are repeated eigenvalues.

stable i.s.L. This holds if the $\mathbf{Re}(\lambda_i) \leq 0$ for all eigenvalues.

unstable This holds if the $\mathbf{Re}(\lambda_i) > 0$ for any eigenvalue.

If we are determining the stability of an equilibrium point in a nonlinear system by looking at the linearized system then we can only say something about the stability of the system if the $\mathbf{Re}(\lambda_i) < 0$ for all eigenvalues. In this case the equilibrium point is asymptotically stable. If we have any zero eigenvalues we can NOT conclude the point is stable i.s.L.

Recall that we can approximate the dynamics near an equilibrium point (with $u = 0$) of a nonlinear system with the linearized approximation

$$\dot{\delta}_x(t) \approx \left.\frac{\partial f}{\partial x}\right|_{x=x_e} \delta_x(t)$$

where $\delta_x(t) \doteq x(t) - x_e$.

Show form of state matrix for a general n-dimensional system.

2 Problem 1

Example of auto regulation in GRNs. Decompose system into a block diagram and describe the difference between the open loop and closed loop behavior.

3 Problem 2

Consider the linear system

$$m_1 \frac{d^2 x_1}{dt^2} = -k_1 x_1 + u$$

Find the stability of the system. We first find the state space form of the system $\dot{x} = Ax + Bu$ where

$$A = \begin{bmatrix} 0 & 1 \\ -k_1/m_1 & 0 \end{bmatrix} \text{ and } B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

We find the eigenvalues have zero real part and are complex conjugates. We determine this by looking at the determinant and trace. We also apply Routh Hurwitz criterion to show the system is not asymptotically stable. Now we sketch what a simulation might look like and the phase plot.

Add a damper to the system. How does this change the behavior of the system? The new system is

$$m\ddot{x} = -kx - b\dot{x}$$

where b is the damping coefficient and k is the spring constant. What is the stability of the new system. Give an example of what a simulation might look like and translate into a phase plot.

4 Problem 3

Consider the linear system

$$m_1 \frac{d^2 x_1}{dt^2} = -k_1 x_1 + u$$
$$m_3 \frac{d^2 x_3}{dt^2} = -k_2 x_3 + u$$

Put the system in state space form. The state equation is

$$\frac{dx}{dt} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -k_1/m_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -k_2/m_2 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1/m_1 \\ 0 \\ 1/m_2 \end{bmatrix} u$$

Discuss what might be possible measured outputs of the system (via what sensors) and determine the corresponding output equation. Discuss what an actuator may look like. Draw a block diagram for such a system, not including a controller yet. Discuss terminology process dynamics, sensing and control law. Find the conditions under which the system is reachable and observable assuming

$$y = [1\,0\,1\,0]$$

(What state is being measured here?)

Let's review definitions of reachability and observability

Reachability A linear dynamical system, $dx/dt = A(t)x(t) + B(t)u(t)$ is said to be (state) controllable at time t_0 if there exists a finite $t_1 > t_0$ such that for any $x(t_0)$ in the state space Σ, there exists an input $u_{[t_0,t_1]}$ that will transfer the state $x(t_0)$ to the state x_1 at time t_1. Otherwise, the system is said to be uncontrollable at t_0. This is equivalent to the matrix below being full rank.

$$P = \begin{bmatrix} B & AB & A^2B & \cdots & A^{n-1}B \end{bmatrix}$$

Observability The linear system

$$\frac{dx(t)}{dt} = Ax(t) + Bu(t) \tag{1}$$

$$y(t) = Cx(t) + Du(t) \tag{2}$$

is said to be observable at t_0 if there exists a finite $t_1 > t_0$ such that for any state x_0 at time t_0, the knowledge of the input $u_{[t_0,t_1]}$ and the output $y_{[t_o,t_1]}$ over the time interval $[t_0, t_1]$ suffices to determine the state x_0. Otherwise, the linear system is said to be unobservable at t_0. This is equivalent to the matrix below being full rank.

$$Q = \begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{n-1} \end{bmatrix}$$

We find that the controllability and observability matrices are

$$P = \begin{bmatrix} 0 & 1/m_1 & 0 & -k_1/m_1^2 \\ 1/m_1 & 0 & -k_1/m_1^2 & 0 \\ 0 & 1/m_2 & 0 & -k_2/m_2^2 \\ 1/m_2 & 0 & -k_2/m_2^2 & 0 \end{bmatrix} \quad Q = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ -k_1/m_1 & 0 & -k_2/m_2 & 0 \\ 0 & -k_1/m_1 & 0 & -k_2/m_2 \end{bmatrix}$$

Now we must determine the rank of the matrix. How to determine whether a matrix is full rank:

- A square matrix is full rank iff the determinant does not equal 0.

- A square matrix can be checked for full rank by looking at either the row vectors or column vectors. Looking at either, the number of vectors in the largest set of linearly independent vectors determines the rank of the matrix. If any one vector is a linear combination of the others then the matrix is not full rank.

In the example problem we can evaluate whether the system is full rank or not using both techniques. So let us do so!

We find the conditions to be $k_1/m_1 \neq k_2/m_2$ for the system to controllable and observable. This can be seen from the determinants of the matrices.

$$\det(P) = \frac{1}{m_1 m_2}\left(\frac{k_2}{m_2}\right)\left(\frac{k_2}{m_2} - \frac{k_1}{m_1}\right)$$

$$\det(Q) = \left(\frac{k_2}{m_2} - \frac{k_1}{m_1}\right)$$

Let's return to Problem 2 with a single mass spring system and assume it is controllable (verify). If our system is controllable then we can find a controller $u = -Kx = -[K_1\, K_2]x$ that stabilizes our system. Find conditions for stability on the controller for this single spring mass system in Problem 2 (without the damper). We find the characteristic equations is

$$s^s + sK_2 + (k_1/m_1 + K_1) = 0$$

so the conditions are $K_2 > 0$ and $(k_1/m_1 + K_1) > 0$.