A PROJECTION OPERATOR APPROACH TO THE OPTIMIZATION OF TRAJECTORY FUNCTIONALS

John Hauser

Electrical and Computer Engineering
University of Colorado
Boulder, CO 80309-0425
hauser@colorado.edu

Abstract: We develop a Newton method for the optimization of trajectory functionals. Through the use of a trajectory tracking nonlinear projection operator, the dynamically constrained optimization problem is converted into an unconstrained problem, making many aspects of the algorithm rather transparent. Examples: first and second order optimality conditions, search direction and step length calculations, update rule—all developed from an unconstrained point of view. Quasi-Newton methods are easily developed as well, allowing straightforward globalization of the Newton method. As all operations are set in an appropriate Banach space, properties such as solution regularity are retained so that implementation decisions (level of discretation, etc.) are based on approximating the solution rather than the problem. Convergence in Banach space is shown to be quadratic as is usual for Newton methods.

Keywords: nonlinear optimal control, trajectory optimization, Newton methods, Banach manifolds, nonlinear projection operators, trajectory manifold.

1. PROBLEM SETTING

We are interested in optimal control problems (OCPs) of the form

\[ \begin{align*}
\text{minimize} & \quad \int_0^T l(\tau, x(\tau), u(\tau)) \, d\tau + m(x(T)) \\
\text{subject to} & \quad \dot{x}(t) = f(x(t), u(t)), \quad x(0) = x_0
\end{align*} \]  

over the class of (essentially) bounded inputs. This problem is often referred to as an unconstrained optimal control problem since, under uniqueness conditions and for fixed \( x_0 \), the state trajectory is completely determined (on its interval of existence) by the choice of control \( x(t) = x(t; u(\cdot)) \) allowing one to remove the dynamic constraint, writing the objective as a function of \( u(\cdot) \) alone. (Such a shooting approach is, of course, not recommended.)

We are mainly interested in objectives and systems that possess a certain degree of smoothness: let \( l(t, x, u), m(x) \), and \( f(x, u) \) be (at least) \( C^3 \) in \( x \) and \( u \) (with \( l(t, x, u) \), e.g., continuous in \( t \)). To ensure that solutions (should they exist) of the optimal control are nice (and somewhat likely), we desire some convexity conditions. We require the set \( f(x, \mathbb{R}^m) \subset \mathbb{R}^n \) to be convex for each \( x \in \mathbb{R}^n \). We also require the pre-Hamiltonian to be strongly convex in \( u \), that is, the map

\[ u \mapsto l(t, x, u) + p^T f(x, u) := H^-(t, x, u, p) \]

is strictly convex for all \( (t, x, p) \in \mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^n \), possessing a second derivative matrix that is uniformly positive definite. This ensures a unique control \( \bar{u}^*(t, x, p) \) that minimizes the pre-Hamiltonian providing a \( C^2 \) (in \( (x, p) \)) Hamiltonian \( H(t, x, p) := H^-(t, x, \bar{u}^*(t, x, p), p) \). This property is satisfied when, e.g., \( f(x, u) \) is affine in \( u \) and \( l \) is quadratic (and positive definite for \( t \in [0, T] \)) in \( u \). To the purpose of existence, we expect the terminal cost \( m \) to be nonnegative (and preferably proper). With sufficient conditions of \( f, l, \) and \( m \), one may guarantee existence of optimal trajectories, see, e.g., (Lee and Markus, 1989; Cesari, 1983). Also of interest here are techniques from the direct methods of the calculus of variations—see (Buttazzo et al., 1998) for an accessible introduction.
2. PROJECTION OPERATOR BASED DESCENT

A trajectory of \( f \) through \( x_0 \) is a bounded curve \( \eta(t) = (x(t), u(t)), \ t \geq 0, \) satisfying \( \dot{x}(t) = f(x(t), u(t)), \ x(0) = x_0. \) Although we will be mostly interested in trajectories on the finite horizon \([0, T]\), it is often useful to consider a finite length trajectory as a portion of one of infinite extent.

Since \( f \) may be inherently unstable, we take a trajectory tracking approach. To this end, suppose that \( \xi(t) = (\alpha(t), \mu(t)), \ t \geq 0, \) is a bounded curve (e.g., an approximate trajectory of \( f \)) and let \( \eta(t) = (x(t), u(t)), \ t \geq 0, \) be the trajectory of \( f \) determined by the nonlinear feedback system

\[
\begin{align*}
\dot{x}(t) &= f(x(t), u(t)), \\
 u(t) &= \mu(t) + K(t)(\alpha(t) - x(t)).
\end{align*}
\]

Under certain conditions on \( f \) and \( K \), this feedback system defines a continuous, nonlinear projection operator

\[
\mathcal{P} : \xi = (\alpha, \mu) \mapsto \eta = (x, u).
\]

First note that, independent of \( K \), if \( \xi \) is a trajectory of \( f \), then \( \xi \) is a fixed point of \( \mathcal{P} \), \( \xi = \mathcal{P}(\xi). \) Now suppose that \( \xi_0 \) is a trajectory of \( f \) of infinite extent and that \( K \) is bounded and such that the above feedback exponentially stabilizes \( \xi_0. \) Then \( \mathcal{P} \) is well defined on an \( L_\infty \) neighborhood of \( \xi_0 \) in the sense that there is an \( \epsilon > 0 \) such that \( \eta = \mathcal{P}(\xi) \) is a (bounded) trajectory of \( f \) for each \( \xi \) with \( \|\xi - \xi_0\|_{L_\infty} < \epsilon. \) In fact, the nonlinear projection operator \( \mathcal{P} \) is \( C^r \) on its domain (including an open neighborhood of \( \xi_0 \)) whenever \( f \) is (Hauser and Meyer, 1998). (By differentiable, we mean Fréchet differentiable with respect to the \( L_\infty \) norm.) Exponential stability plays an important role in making this operator continuous. Further properties of \( \mathcal{P} \) can be used to show that the set of exponentially stabilizable trajectories of \( f \) is a Banach manifold (Hauser and Meyer, 1998). Using \( T \) to denote the trajectory manifold, we see that \( \xi \in T \) if and only if \( \xi = \mathcal{P}(\xi). \) Note also that \( \mathcal{P} \) is a projection since \( \mathcal{P} = \mathcal{P} \circ \mathcal{P} \) on its domain.

The trajectory tracking projection operator \( \mathcal{P} \) is also useful in the consideration of trajectories on a finite interval. In this case, \( K \) is chosen to make the modulus of continuity of \( \mathcal{P} \) reasonably small. Indeed, unless the feedback system possesses a stability-like property, the resulting trajectories may grow so quickly that they are, for all practical purposes, unbounded. In such a case, the domain of the projection operator will be so small as to be useless for computations (an example of the instability of shooting).

With a suitable projection operator in hand, we can represent the optimal control problem as a constrained optimization problem.

Let \( X \) denote the closed subspace of \( L^{n+m}_\infty[0, T] \) of curves \( \zeta = (\beta, \nu) \) with continuous \( \beta, \beta(0) = 0, \) and bounded \( \nu. \) Equipped with the norm \( \|\zeta\|_{X} = \|\zeta\|_{L_\infty}, \) \( X \) is a Banach space. Define \( \pi_1 := [I \ 0] \) and \( \pi_2 := [0 \ I] \) so that \( \beta = \pi_1 \zeta \) and \( \nu = \pi_2 \zeta. \) Trajectories of \( f \) through \( x_0 \) belong to the affine subspace \( \tilde{X} := (x_0, 0) + X. \) Defining the functional

\[
\begin{align*}
\tilde{h}(\zeta) := \int_{0}^{T} l(\tau, \alpha(\tau), \mu(\tau)) \ d\tau + m(\alpha(T))
\end{align*}
\]

for curves \( \zeta = (\alpha, \mu) \in \tilde{X} \), we see that the optimal control problem (1) is equivalent to the constrained optimization problems

\[
\min_{\zeta \in \mathcal{P}(\mathcal{P}(\xi))} \tilde{h}(\zeta) = \min_{\zeta \in \mathcal{P}(\xi)} \tilde{h}(\zeta)
\]

where the constraint set \( T \) is a Banach submanifold of \( \tilde{X}. \) Defining

\[
\tilde{g}(\zeta) := h(\mathcal{P}(\xi))
\]

for \( \zeta \in \mathcal{U} \subset \tilde{X} \) with \( \mathcal{P}(\mathcal{U}) \subset \mathcal{U} \subset \text{dom} \mathcal{P}, \) we see that the optimization problems

\[
\min_{\zeta \in \mathcal{P}(\mathcal{U})} \tilde{h}(\zeta) \quad \text{and} \quad \min_{\zeta \in \mathcal{P}(\mathcal{U})} \tilde{g}(\zeta)
\]

are equivalent in the following sense. If \( \xi^* \in T \cap \mathcal{U} \) is a constrained local minimum of \( \tilde{h}, \) then it is an unconstrained local minimum of \( \tilde{g}. \) If \( \xi^*+ \in \mathcal{U} \) is an unconstrained local minimum of \( g \) in \( \mathcal{U}, \) then \( \xi^* = \mathcal{P}(\xi^*) \) is a constrained local minimum of \( T. \)

This observation is the basis for the development of a family of quasi-Newton descent methods for the optimization of \( \tilde{h} \) over \( T. \)

The projection operator \( \mathcal{P} \) provides a convenient parametrization of the trajectories in the neighborhood of a given trajectory. Indeed, the tangent space \( T_{\xi} T \) of bounded trajectories of the linearization of \( \dot{x} = f(x, u) \) about \( \xi \in T \) can be used to parametrize all nearby trajectories (Hauser and Meyer, 1998). That is, given \( \xi \in T, \) there is an \( \epsilon > 0 \) such that, for each \( \eta \in T \) with \( \|\eta - \xi\| < \epsilon \) there is a unique \( \zeta \in T_{\xi} T \) such that \( \eta = \mathcal{P}(\xi + \zeta). \) (Of course, there are many other curves \( \xi \in U \) of the trajectory \( \eta = \mathcal{P}(\xi). \) This robust representation of \( \eta \) is ideally suited to numerical computations since the approximation errors introduced by discretization in time and quantization in space are kept small by the stabilizing effect of the feedback. In contrast, if \( f \) is unstable, it is easy to find multiple trajectories for which the initial condition and control trajectories are the same to machine precision. A suitable feedback gain \( K \) may be constructed by, for example, solving a finite horizon linear regulator problem (Anderson and Moore, 1990) about the trajectory \( \eta. \)
is the bounded linear projection operator defined by linearizing (2) about $\xi$ and that $\zeta \in T_\xi \mathcal{T}$ if and only if $\zeta = D\mathcal{P}(\xi) \cdot \zeta$.

We propose the following Newton method for the optimization of trajectory functionals.

**Algorithm (projection operator Newton method)**

given initial trajectory $\xi_0 \in \mathcal{T}$

for $i = 0, 1, 2, \ldots$

redesign feedback $K$ if desired/needed search direction

$$
\zeta_i = \arg \min_{\zeta \in \mathcal{T}_\xi \mathcal{T}} Dg(\xi_i) \cdot \zeta + \frac{1}{2} D^2 g(\xi_i) \cdot (\zeta, \zeta)
$$

(3)

step size

$$
\gamma_i = \arg \min_{\gamma \in (0, 1)} g(\xi_i + \gamma \zeta_i)
$$

update

$$
\xi_{i+1} = \mathcal{P}(\xi_i + \gamma_i \zeta_i)
$$

(4)

end

This algorithm is quite similar to the usual Newton method for unconstrained optimization of a function $g(\cdot)$ (e.g., in finite dimensions). As usual, the second order Taylor polynomial is used as a quadratic model function for determining a descent direction. A pure Newton method would, of course, use a fixed step size of $\gamma_i = 1$ — the line search is common for expanding the region of convergence. The key differences are that 1) the search direction minimization (3) is performed on the tangent space to the trajectory manifold and 2) the update (4) projects each iterate on to the trajectory manifold. The algorithm is easily generalized (or globalized) by replacing the Newton direction calculation (3) by a quasi-Newton search direction calculation

$$
\zeta_i = \arg \min_{\zeta \in \mathcal{T}_\xi \mathcal{T}} Dg(\xi_i) \cdot \zeta + \frac{1}{2} D^2 g(\xi_i) \cdot (\zeta, \zeta)
$$

(5)

where $g(\xi_i)$ is a suitable positive definite (to be defined below) approximation to $D^2 g(\xi_i)$.

The remainder of the paper is devoted to demonstrating that

- the search direction subproblems (3) and (5) are well defined (with suitable continuity properties) provided that the quadratic forms are $L_2$ positive definite on the tangent space to $\mathcal{T}$ and
- the Newton algorithm (with $\zeta_i \equiv 1$) provides (local) quadratic convergence to a local minimum satisfying second order sufficiency conditions.

We will see that linear projection operator $D\mathcal{P}(\xi)$ will play a key role in this endeavor.

3. **PROJECTION OPERATOR CALCULATIONS**

We provide formulas for $\mathcal{P}$ derivatives to third order. The theoretical development of these forms is the bounded linear projection operator defined by linearizing (2) about $\xi$ and that $\zeta \in T_\xi \mathcal{T}$ if and only if $\zeta = D\mathcal{P}(\xi) \cdot \zeta$.

As expected, the derivative of the projection operator $\mathcal{P} : \tilde{X} \rightarrow \mathcal{T} \subset \tilde{X}$ is the linear projection operator $D\mathcal{P}(\xi) : \mathcal{X} \rightarrow \mathcal{X}$ given by the standard linearization. That is, we can compute $\gamma = (z, v) = D\mathcal{P}(\xi) \cdot \zeta$, with $\xi \in (\alpha, \mu) \in \tilde{X}$, $\eta = (x, u) = \mathcal{P}(\xi) \in \mathcal{T}$, and $\zeta \in (\beta, \nu) \in \mathcal{X}$, using $\dot{z}(t) = A(\eta(t)) z(t) + B(\eta(t)) v(t)$, $z(0) = 0$, $v(t) = \nu(t) + K(t)(\beta(t) - z(t))$

where $A(\eta(t)) = D_2 f(x(t), u(t))$ and $B(\eta(t)) = D_2 f(x(t), u(t))$. Using $\Phi_2(t, \tau)$ to denote the state transition matrix of the closed loop dynamics matrix $A_c(\mathcal{P}(\xi)(t)) = A_c(\eta(t)) = A(\eta(t)) - B(\eta(t)) K(t)$, we obtain

$$
\gamma(t) = (D\mathcal{P}(\xi) \cdot \zeta)(t) = \left[ \begin{array}{cc} 0 & 0 \\ K(t) & I \end{array} \right] \zeta(t)
$$

(7)

$$
+ \left[ I - K(t) \right] \int_0^t \Phi_2(t, \tau) B(\eta(\tau)) [-K(\tau) I] \zeta(\tau) d\tau
$$

Note that $D\mathcal{P}(\xi)$ is a linear projection operator: $D\mathcal{P}(\xi) \cdot \zeta = D\mathcal{P}(\xi) \cdot D\mathcal{P}(\xi) \cdot \zeta$ for all $\zeta \in \mathcal{X}$.

Note also that the character of $D\mathcal{P}(\xi)$ depends only on the trajectory $\eta = \mathcal{P}(\xi) \in \mathcal{T}$ and not on the particular $\xi \in \mathcal{P}^{-1}(\eta)$. The fact that $D\mathcal{P}(\xi)$ is a continuous linear projection ensures that $X_\xi := T_\xi \mathcal{T}$ is a split subspace of $X$ (i.e., $X_\xi$ is closed with closed complement).

Higher order derivatives of $\mathcal{P}$ (with respect to $L_\infty$) may be obtained using the chain rule to differentiate (7). The dependence of $\Phi_2(t, \tau)$ on the trajectory $\eta = \mathcal{P}(\xi)$ is taken into account using of the general formula (with slightly different notation)

$$
(D\psi_r(\xi) \cdot \zeta)(t) = \int_0^t \Phi_2(t, s) [D(A(\xi(s)) \cdot \zeta(s))] \Phi_2(s, \tau) ds
$$

(8)

where $\psi_r(\xi)(t) = \Phi_2(t, \tau)$ is the state transition matrix corresponding to the system matrix $A(\xi(t))$. Thus, one would apply the chain rule to $\psi_r(\mathcal{P}(\xi)(t))(t) = \Phi_2(t, \tau)$, using $A_c(\mathcal{P}(\xi)(t))$ as the system matrix. Despite the tedium involved in obtaining higher order derivatives in this fashion, the resulting expressions are quite simple indeed.

Letting $\gamma_i = D\mathcal{P}(\xi) \cdot \zeta_i$ and $\omega_{i,j} = D^2 \mathcal{P}(\xi) \cdot (\zeta_i, \zeta_j)$, one obtains the symmetric expressions

$$
\omega_{i,j}(t) = (D^2 \mathcal{P}(\xi) \cdot (\zeta_i, \zeta_j))(t)
$$

(9)

$$
- \left[ I - \frac{t}{2} \right] \int_0^t \Phi_2(t, \tau) D^2 f(\psi(\tau))(\zeta_i(\tau), \zeta_j(\tau)) d\tau
$$
(D³P(ξ) · (ξ₁, ξ₂, ξ₃))(t) =
\[\begin{bmatrix}
I & 0 & 0 \\
-\frac{1}{{K(t)}} & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
P_\epsilon(t, \tau) \cdot \{ D³f(ξ(\tau)) · (ξ₁(\tau), ξ₂(\tau), ξ₃(\tau)) \\
D²f(ξ(\tau)) · (ξ₁(\tau), ξ₂(\tau)) \\
D²f(ξ(\tau)) · (ξ₁(\tau))
\end{bmatrix}
\begin{bmatrix}
(ξ₁(\tau), ξ₂(\tau), ξ₃(\tau)) \\
(ξ₁(\tau), ξ₂(\tau)) \\
(ξ₁(\tau))
\end{bmatrix}
\, dτ
\]

As in the first derivative case, the projection property of \( P \) implies that the higher derivative expressions depend only on the trajectory \( η = P(ξ) \) and the linear projection of its arguments onto the tangent space \( \mathcal{T}_T = T_0 \mathcal{T} \) so that, for example,

\[D²P(ξ) · (ξ₁, ξ₂) = D²P(ξ) · (D²P(ξ) · ξ₁, D²P(ξ) · ξ₂) (10)\]

for all \( ξ₁, ξ₂ \in \mathcal{X} \).

Now, by definition, \( D³P(ξ) \) is a mapping from \( X \) to \( X \). However, direct reference to the formula (7) indicates that there is little to keep us from attempting to apply, at least formally, the operator \( D³P(ξ) \) to curves \( η \in Y := L^{n+m}_{2+}[0, T] \). Noting that \( X \) is continuously imbedded (even densely imbedded) in \( Y \), we can consider the resulting operator, denoted \( D³P(ξ)_Y \), to be some type of extension of \( D³P(ξ) \) to \( Y \) (remembering, of course, that \( X \) and \( Y \) have different norms). The following result will prove invaluable in determining the nature of the solutions to (3) and (5).

**Lemma 3.1.** The continuous linear operator \( D³P(ξ) : X \to Y \) (for any bounded \( K \)) defines a continuous linear operator \( D³P(ξ)_Y : Y \to Y \).

**Proof:** Let \( η \in Y \) and compute \( γ \) using (7). By the boundedness of \( Φ, B, \) and \( K \), we see that \( \|γ(t)\| \leq c_1 |ξ|_{L₁} + c₂ |ξ(t)| \) for some constants \( c₁, c₂ \). Now, since Hölder’s inequality (or the Cauchy-Schwarz inequality) implies that \( \|ξ\|_{L₁} \leq \sqrt{T} \|ξ\|_{L₂} \), it follows (using the triangle inequality) that \( \|ξ\|_{L₂} \leq c_3 \|ξ\|_{L₂} \) for some \( c₃ \).

The range of the projection operator \( D³P(ξ)_Y \), denoted \( Yₚ \), is the closed subset of \( Y \) analogous to \( Xₚ = T_0 \mathcal{T} \). Note that each curve \( η \in Yₚ \) is such that the state portion of the curve \( p(ξ) \) is continuous in time (with \( \|p(ξ)\|_{L₂} \leq c₄ \|π₂(ξ)\|_{L₂} \) for some \( c₄ \)), allowing well defined point evaluations, e.g., \( π₁(ξ(T)) \). It is worth pointing out that the higher derivatives \( D³kP(ξ) \) may also be extended to multilinear operators on \( Y \) since, as in (10), each of the arguments \( ξᵢ \) is first filtered by \( D³P(ξ) \).

The linear projection operator \( D³P(ξ) \) can also be used to highlight the nature of the linear and multilinear functionals that arise in this work, namely the derivatives of the optimization objective \( g \). In that \( X \) is the cartesian product of a subspace of \( G \) and \( L_{∞} \), it is clear that the space of bounded linear functionals is indeed. We are fortunate that the set of functionals of interest to us is much less rich.

The following result will be especially useful in the study of convergence of the above Newton’s method.

**Proposition 3.2.** Let \( ξ \in \mathcal{T} \) and \( δ, ζ \in Xₚ = T_0 \mathcal{T} \). For each \( k \), such that \( D³kP(ξ) \) is defined, there is an \( r_k \in L_{∞} \) and a \( c_k < ∞ \) such that

\[D³kP(ξ) · (δ, ζ) = \int_{0}^{T} r_k(τ)T₂(τ)ζ(τ)dτ (11)\]

with \( \|r_k\|_{L₂} ≤ c_k \|δ\|^k_{X₂} \). (12)

Note that we get much stronger bounds for this class of functionals than normally expected for linear functionals in Banach space.

**Proof:** Consider first the case \( k = 1 \). We have

\[D³g(ξ) · ζ = D³g(ξ) · γ = \int_{0}^{T} D³g(ξ) · (ξ(τ), ζ(τ))dτ + Dm(π₁(ξ(T))) · π₁(γ(τ)) \cdot (11)\]

Set \( a(γ) := D³g(ξ) · (γ(τ)) \) and \( aₗ := Dm(π₁(ξ(T))) \). Using (7) and changing the order of integration, we find that \( D³g(ξ) · ζ = (r, ζ)ₗ \) where

\[r(τ) = \left[ \begin{array}{c}
K(τ)² \\\nI
\end{array} \right] \left[ \begin{array}{c}
B(ξ(τ))P(τ) + π₂(aₗ(τ)) \end{array} \right] (13)\]

That \( c₁ := \|r₁\|_{L₂} < ∞ \) follows easily from the boundedness of \( K \) and the objects based on them including \( a, Φ, \) and \( B \). In fact, \( \|r₁\|_{L₂} ≤ c \max\{\|a\|_{L₂}, \|a₁\| \} \) for some \( c < ∞ \).

The case \( k = 2 \) will help establish the general pattern. It is easy to see that, for \( ξ \in \mathcal{T} \) and \( δ, ζ \in Xₚ \), \( D³²P(ξ) · (δ, ζ) = D³²g(ξ) · (δ, ζ) + D³g(ξ) · (δ, ζ) \cdot D³²P(ξ) \cdot (δ, ζ) \). Using the above adjoint technique with \( p \) given by (14), one finds that

\[D³²g(ξ) · (δ, ζ) = \int_{0}^{T} \left[ W(ξ(τ))ζ(τ) + D²g(ξ(τ))ζ(τ) \right]ds (15)\]

so that the second derivative is given by

\[D³²P(ξ) · (δ, ζ) = \int_{0}^{T} \left[ D²g(ξ(τ))ζ(τ) + D²g(ξ(τ))ζ(τ) + D²g(ξ(τ))ζ(τ) \right]ds + (π₁(δ(T)))² \cdot (π₁(ξ(T)))² \]

where \( W(t) = [w_{ij}(t)] \) is the bounded symmetric \( (n + m) \times (n + m) \) matrix with entries given by

\[w_{ij}(t) = \frac{∂²}{∂t²} \left( \frac{∂}{∂t} \cdot (ξ(t), ζ(t)) \right) \cdot (ξ(t), ζ(t)) \]
and \( P_1 \) is the symmetric \( n \times n \) matrix representing the bilinear operator \( D^2m(\pi_1(\xi(T))) \). Defining \( b(\tau) = W(\tau)\delta(\tau) \) and \( b_1 = P_1 \pi_1(\delta(\tau)) \) and noting that \( \gamma = \zeta = DP(\xi) \cdot \zeta \), we find that \( D^2g(\xi) \cdot (\delta, \zeta) = (r_2, \zeta)_{L_2} \) where \( r_2 \) is determined by formulas analogous to (13), (14) with \( b \) and \( b_1 \) replacing \( a \) and \( a_1 \). The existence of \( c_2 < \infty \) such that \( \|r_2\| \leq c_2\|\delta\| \) follows immediately. The cases \( k > 2 \) follow in a similar manner.

The representation (11) provides another path for expanding the domain of these linear functionals (of \( \zeta \)) from \( X \) to \( Y \). Moreover, expressions such (15) for \( D^2g(\xi) \) may obviously be evaluated on \( \delta, \zeta \in Y_\xi \) (rather than \( \delta, \zeta \in X_\xi \)). The estimate (12) with \( k = 3 \) will be used in the demonstration of quadratic convergence.

4. QUADRATIC MINIMIZATION AND INVERSION

The search direction subproblem ((3) or (5)) requires the minimization of a quadratic model function \( Dh(\zeta) \cdot \zeta + \frac{1}{2}q(\zeta) \cdot (\zeta, \zeta) \) over the Banach space \( X_\xi = T_\xi T \). As with the formula for \( D^2g(\xi) \), (15), the quadratic functional \( q(\xi) \) is chosen to be of the form

\[
q(\xi) \cdot (\zeta, \zeta) = \int_0^T \zeta^T W(\tau) \zeta(\tau) d\tau + \beta(T)^T P_1 \beta(T)
\]

where \( W(\tau) = W(\tau)^T = \begin{bmatrix} Q(\tau) & S(\tau) \\ S(\tau)^T & R(\tau) \end{bmatrix} \) is bounded and \( R \) is uniformly positive definite (\( R(\tau) \geq r_0I \) for some \( r_0 > 0 \)).

Clearly, if the quadratic functional was such that

\[
q(\xi) \cdot (\zeta, \zeta) \geq 0
\]

for \( \zeta \in X_\xi \), the desired minimum would exist for then the linear map

\[
q(\xi) : X_\xi \to X_\xi^*: \zeta \mapsto q(\xi) \cdot (\zeta, \cdot)
\]

would be invertible. Unfortunately, a functional of the form (16) cannot be strongly positive definite with respect to the \( L_\infty \) norm.

It is, however, possible for such quadratic functionals to satisfy a bound of the form

\[
q(\xi) \cdot (\zeta, \zeta) \geq 0
\]

for \( \zeta \in X_\xi \). Moreover, \( q(\xi) \) is also a well defined quadratic functional on the Hilbert space \( Y_\xi \). The following result is well known (Maurer, 1981) (cf. (Bryson and Ho, 1969)).

**Proposition 4.1.** If the Riccati equation

\[
\dot{P} + A^T P + P A - B R^{-1} B^T P + Q = 0, \quad P(T) = P_1,
\]

where \( A = A_T) \) and \( B R^{-1} B^T \) are invertible, then the functional \( q(\xi) \) is strongly positive on the Hilbert space \( Y_\xi \). That is, there is a \( q_0 > 0 \) such that

\[
q(\xi) \cdot (\zeta, \zeta) \geq q_0 \|\zeta\|_{L_2}^2
\]

for all \( \zeta \in Y_\xi \) and hence for all \( X_\xi = T_\xi T \).

Note that, although the conditions \( W(t) \geq w_0I, \) \( P_1 > 0, \) are sufficient to ensure the strong \( L_2 \) positivity of \( q(\xi) \), much less is needed. Boundedness of \( W \) implies that \( q(\xi) \) is a bounded bilinear operator (in both \( L_2 \) and \( L_\infty \)). In particular, there is a \( q_1 < \infty \) such that

\[
q_0 \|\zeta\|_{L_2}^2 \leq q(\xi) \cdot (\zeta, \zeta) \leq q_1 \|\zeta\|_{L_\infty}^2
\]

for all \( \zeta \in Y_\xi \).

**Proposition 4.2.** Suppose that \( q(\xi) \) is a strongly positive quadratic functional satisfying the Riccati boundedness condition and let \( \bar{r} \) be a linear functional of the form

\[
\bar{r} \cdot \zeta = \int_0^T a(\tau)^T \zeta(\tau) d\tau + a_1^T \pi_1(\zeta(T))
\]

where \( a \in L_\infty \). The quadratic minimization problem

\[
\min_{\zeta \in Y_\xi} \bar{r} \cdot \zeta + \frac{1}{2}q(\xi) \cdot (\zeta, \zeta)
\]

has a unique solution \( \zeta^* \) belonging to \( X_\xi = T_\xi T \) and satisfying

\[
\|\zeta^*\|_{L_2} \leq \frac{1}{\sqrt{q_0}} \|\bar{r}\|_{Y_\xi^*}
\]

The optimal descent satisfies

\[
\bar{r} \cdot \zeta^* \leq -q(\xi) \cdot (\zeta^*, \zeta^*) \leq -\frac{q_1}{q_0} \|\bar{r}\|_{L_\infty}^2
\]

Moreover, there is a \( q_2 < \infty \) such that

\[
\|\zeta^*\|_{L_\infty} \leq q_2 \max\{|a|_{L_\infty}, |a_1|\}
\]

**Proof:** By hypothesis, \( q(\xi) \) is symmetric, bounded, and strongly positive on the Hilbert space \( Y_\xi \) and \( \bar{r} \) is a continuous linear functional on \( Y_\xi \). It follows (Zeidler, 1995, Theorem 2.A) that the variational minimization problem (18) has a unique solution (in \( Y_\xi \)) satisfying the equality in (20).

By the Riesz representation theorem, there is an \( r \in Y_\xi \) with \( \|r\|_{Y*} = \|\bar{r}\|_{Y*} \) such that \( \bar{r} \cdot \zeta = (r, \zeta)_{Y*} \) for all \( \zeta \in Y_\xi \). Since \( \zeta^* \) is optimal, we have, using \( \zeta = -\varepsilon r \) with \( \varepsilon = 1/q_1 \),

\[
-q(\xi) \cdot (\zeta^*, \zeta^*) = 2\bar{r} \cdot \zeta^* + q(\xi) \cdot (\zeta^*, \zeta^*)
\]

\[
\leq -2\varepsilon\|r\|_{Y*}^2 + \varepsilon^2 q_1\|r\|_{Y*}^2 = -\frac{q_1}{q_0}\|r\|_{Y*}^2
\]

proving the inequality of (20). A further application of the upper bound on \( q(\xi) \) yields (19).

Problem (18) is easily seen to be a linear quadratic optimal control problem, solvable by standard techniques (Anderson and Moore, 1990). Its solution can be shown to be bounded by direct calculation.
the special feedback gain \( K_o(t) := R(t)^{-1}(S(t) + B(\xi(t))^TP(t)) \) and let \( \Phi_{eo} \) denote the associated closed loop state transition matrix. The optimal solution is then given by

\[
\zeta(t) = \left[ \begin{array}{c} I \\ -K_o(t) \end{array} \right] t \int_0^t \Phi_{eo}(t, \tau) B(\xi(\tau)) R(\tau)^{-1} \{ -B(\xi(\tau))^TP_o(\tau) - \pi_2 a(\tau) \} d\tau + \left[ \begin{array}{c} 0 \\ R(t)^{-1} \end{array} \right] \{ -B(\xi(t))^TP_o(t) - \pi_2 a(t) \}
\]

with

\[
p_o(t) = \Phi_{eo}(T, t)^T a_1 + \int_t^T \Phi_{eo}(s, t)^T \left[ I - K_o(s)^T \right] a(s) \, ds.
\]

The bound (21) follows easily.

5. LOCAL CONVERGENCE

The local equivalence of \( \min_{\xi \in T} h(\xi) \) and \( \min_{\xi \in U} g(\xi) \) leads to very simple optimality conditions. For example, first order necessary condition is simply: If \( \xi^* \) is a (local) minimum of \( h \) over \( T \), then \( Dg(\xi^*) \cdot \zeta = 0 \) for all \( \zeta \in X \) (conveniently written as \( Dg(\xi^*) = 0 \)). Exploiting the linear projection operator present in \( g \), we see that this is equivalent to the usual condition \( Dh(\xi^*) \cdot \zeta = 0 \) for all \( \zeta \in X_{\xi^*} = T_{\xi^*} T \). The second order sufficiency condition also arises naturally (cf. (Ioffe, 1979; Maurer, 1981)): If \( \xi^* \) is such that \( Dg(\xi^*) \cdot \zeta = 0 \) for all \( \zeta \in X \) and \( D^2g(\xi^*) \cdot \zeta \geq ||\zeta||_2^2 / \lambda_2 \) for all \( \zeta \in X_{\xi^*} \), then \( \xi^* \) is an isolated local minimum.

Proposition 5.1. The projection operator based Newton method (with unit step size) provides locally quadratic convergence.

Proof: Let \( \xi^* \) be a local minimum satisfying the second order sufficiency optimality condition. We have

\[
0 = Dg(\xi^*) = Dg(\xi^*) + D^2g(\xi^*) \cdot (\xi^* - \xi_i) + \int_0^1 (1-s) D^3g(\xi_i + s(\xi^* - \xi_i)) ds \cdot (\xi^* - \xi_i, \xi^* - \xi_i) = D^2g(\xi^*) \cdot (\xi^* - (\xi_i + \zeta_i)) + R_3(\xi_i) \cdot (\xi^* - \xi_i, \xi^* - \xi_i)
\]

where the Newton step \( \zeta_i \) is such that \( Dg(\xi_i) = -D^2g(\xi_i) \cdot \zeta_i \) and \( R_3(\xi_i) \) is the trilinear integral remainder expression. Now, since

\[
D^2g(\xi_i) \cdot (\xi^* - (\xi_i + \zeta_i), \gamma) = D^2g(\xi_i) \cdot (\xi^* - (\xi_i + \zeta_i), DP(\xi_i) \cdot \gamma)
\]

for all \( \gamma \in X \), it must also be that

\[
R_3(\xi_i) \cdot (\xi^* - \xi_i, \xi^* - \xi_i, \gamma)
\]

for all \( \gamma \in X \). Thus, by (an extended version of) proposition 3.2, there is a bounded \( r_3 \) and a \( c_3 < \infty \) such that

\[
|R_3(\xi_i) \cdot (\xi^* - \xi_i, \xi^* - \xi_i, \gamma)| \leq r_3 \cdot c_3 \cdot ||\xi^* - \xi_i||^2_X
\]

for all \( \gamma \in X \) with \( ||\gamma||_X \leq c_3 \cdot ||\xi^* - \xi_i||^2_X \).

Although, as stated \( r_3 \) depends on \( \xi_i \), we can, by continuity, find a \( c_3 \) that works for every \( \xi_i \) in a closed \( \epsilon \) neighborhood about \( \xi^* \). Using proposition 4.2 to solve

\[
D^2g(\xi_i) \cdot (\xi^* - (\xi_i + \zeta_i), \gamma) = r_3 \cdot c_3 \cdot ||\xi^* - \xi_i||^2_X
\]

for some \( k < \infty \) and all \( ||\xi_i - \xi^*||_X \leq \epsilon \). After further manipulations, one can show that there is a \( k_1 < \infty \) and an \( \epsilon_1 > 0 \) such that, for all \( ||\xi_i - \xi^*||_X \leq \epsilon_1 \),

\[
||\xi_i - \xi^*||_X \leq k_1 \cdot ||\xi^* - \xi_i||^2_X
\]

demonstrating local quadratic convergence.

6. REFERENCES


