Controls Primer, Lect. 3
25 Sept 2001

So far:
- intro. to control thy,
- basic concepts of feedback control
- several apps. & examples
Outline:

- defin. of stability / classes of stab.
- Lyapunov's direct stab. thm.
- examples
- ...

Types of stab:

- BIBO (bounded inputs $\Rightarrow$ bounded outputs)
- input-to-state, input-to-output
- stab. of systems, equilibrium pts., trajectories, ...
Consider:
\[
\dot{x} = f(x,t), \quad \forall t \geq t_0 \\
x(t_0) = x_0, \quad x \in \mathbb{R}^n
\]

Stab. of an eq. pt. of the system, i.e. \( x^* \) s.t. \( f(x^*,t) = 0 \)

W/o loss of generality, assume \( x^* = 0 \). To see:

\[
\dot{z} = g(z,t), \quad \forall t \geq t_0 \\
z(t_0) = z_0, \quad z \in \mathbb{R}^n \quad \text{where } z^* = z_0 \text{ is eq. pt.}
\]

Let \( x = z - z_0 \) \( \Rightarrow \frac{\dot{x}}{z} = g(x + z, t) \quad \overset{a}{=} \quad f(x,t) \)
**Defn 4.1** The eq. $\dot{x} = 0$ is stable (in the sense of Lyapunov) at $t = t_0$. 

\[ \forall \varepsilon > 0 \exists \delta(t_0, \varepsilon) > 0 \text{ s.t.} \]

\[ \| x(t_0) \| < \delta \implies \| x(t) \| < \varepsilon \quad \forall t \geq t_0 \]

**Interpretation**
Def. A: A solution \( x(t; \xi) \) of (1) is (Lyap.) stable at \( t_0 \) if \( \forall \varepsilon > 0, \exists \delta = \delta(t_0, \varepsilon) \) s.t. \((\xi', \varepsilon) \in \mathbb{R}^n) \implies ||\xi - \xi'|| < \delta \implies ||x(t; \xi) - x(t; \xi')|| \leq \varepsilon \quad \forall t.

Interpretation \( n = 4 \):

Consider \( \xi' = 0 \) (eq.)
Defn. B: The soln. $x(t;\varepsilon)$ of $(x)$ is (yep!) asymptotically stable at $t=t_0$ if:

1. it is stable (is L) at $t_0$, and
2. \[ \exists \gamma = \gamma(t_0) \; s.t. \; \| x_0 - \varepsilon \| < \gamma \Rightarrow \lim_{t \to \infty} \| x(t;\varepsilon) \| = 0 \]

\[ \ddot{x} = f(x,t) \]

Solution asymptotic stability
\[ x' = -t \]

\[ x'' = e^{-t} x \]

**Def**: Stable Solns are called uniformly stable if \( \delta(t) \), \( \gamma(t) \) can be chosen which do not depend on \( t_0 \).

4.1 A \[ \delta(t_0, \varepsilon) \] instead of \( \delta(t_0) \), have \( \delta(c) \)

4.2 \[ \gamma(t_0) \]

**Note**: \[ x' = f(x) \] (not \( f(x, t) \))

\[ \Rightarrow \text{all forms of Lyap. stab. are uniform} \]
Defn: \text{Soln} \overset{\text{globally asympt. stable (GAS)}}{\Rightarrow} \text{soln is asympt. stable} \iff \dot{y}(t) \in \mathbb{R}^n \quad \forall t \geq t_0 \in \mathbb{R}^n
Thm (Lyapunov's direct method, aut. systems)

Let \( x^* = 0 \) be an eq. pt. for \( \dot{x} = f(x) \), \( 0 \in \Omega \subset \mathbb{R}^n \), and
\( V: \Omega \to \mathbb{R} \) be a ctsly. diffable fn.

If \( \exists \) a \( V(x) \) satisfying

(i) \( V(0) = 0 \), and
(ii) \( V(x) > 0 \) in \( \Omega - \{0\} \), and
(iii) \( \dot{V}(x) \leq 0 \) in \( \Omega \),

then \( x = 0 \) is stable (i.e.), \( \dot{V}(x(t)) = \frac{\partial V}{\partial x} \cdot \dot{x} \leq 0 \).

If \( \exists \) a \( V(x) \) satisfying (i),(ii),(iii), and
(iv) \( \dot{V}(x) < 0 \) in \( \Omega - \{0\} \),

then \( x = 0 \) is asymptotically stable.
Example:

\[
\dot{x} = A x = f(x) \\
V(x) = x' P x
\]

Linearized about \( \varepsilon = (0,0) \):

\[
\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} -x - 2y \\ xy - y' \end{pmatrix}
\]

Try:

\[
V(x) = \frac{1}{2} (x^2 + ay^2)
\]

\( a > 0 \):

(i) \( V \) is positive definite

(ii) \( \frac{d}{dt} V(x) = \frac{d}{dt} V(\varepsilon) = 0 \)

(iii) \( \frac{d}{dt} V(\varepsilon) = \frac{d}{dt} V(x(t)) = 2x \dot{x} + 2ay \dot{y} \)

Let \( a > 0 \):

\[
\dot{V} = - (x^2 + 2y y')
\]

(iii) \( \checkmark \)

(iv) \( \checkmark \)