Goals:
• Describe the use of Kalman filtering for sensor fusion
• Give extensions of Kalman filters to nonlinear systems, discrete time systems

Reading:
• Friedland, Chapter 11
Kalman Filter

System dynamics: linear process + Gaussian white noise

\[
\begin{align*}
\dot{x} &= Ax + Bu + Fv \\
y &= Cx + w
\end{align*}
\]

\[
E\{v(s)v^T(t)\} = Q(t)\delta(t - s) \\
E\{w(s)w^T(t)\} = R(t)\delta(t - s)
\]

Estimator: prediction + correction

\[
\begin{align*}
\hat{x} &= A\hat{x} + Bu + L(y - C\hat{x}) \\
L(t) &= P(t)C^T R^{-1} \\
P(t) &= E\{(x(t) - \hat{x}(t))(x(t) - \hat{x}(t))^T\}
\end{align*}
\]

Covariance update

\[
\begin{align*}
\dot{P} &= AP + PA^T - PC^T R^{-1}(t) CP + FQ(t)F^T \\
P(0) &= E\{x(0)x^T(0)\}
\end{align*}
\]
Variation #1: Sensor Fusion

What happens if we have redundant sensors?

- Kalman filter “fuses” data measurements according to covariance

\[ \dot{x} = A\hat{x} + Bu + L(y - C\hat{x}) \]

\[ L(t) = P(t)C^TR^{-1} \]

\[ P(t) = E\{(x(t) - \hat{x}(t))(x(t) - \hat{x}(t))^T\} \]

- Assume R is diagonal, expand out gain:

\[ L(t) = P(t)C^T \begin{bmatrix} R_{11}^{-1} & \cdots \\ \cdots & \cdots \end{bmatrix} \]

\[ \text{\( R_{11} \text{ large } \Rightarrow \text{ smaller effect} \) \n  \text{\( P \text{ small } \Rightarrow \text{ don't rely on sensors (state is accurate) } \) \n  \text{\( R \text{ small decreases uncertainty} \) \n  \text{\( P \text{ evolves according to nominal dynamics} \) \n  \text{\( \text{Steady state (ARE): optimal balance of dynamics and uncertainty} \) \n
6 Feb 06    R. M. Murray, Caltech
Variation #2: Extended Kalman Filter (EKF)

Consider a *nonlinear* system

\[
\dot{x} = f(x, u, v) \quad x \in \mathbb{R}^n, u \in \mathbb{R}^m \\
y = Cx + w \quad v, w \text{ Gaussian white noise processes with covariance matrices } Q \text{ and } R.
\]

Form estimator using nonlinear model + linear feedback

\[
\hat{x} = f(\hat{x}, u, 0) + L(y - C\hat{x})
\]

Compute estimator gain based on linearization at current estimated state:

\[
\begin{align*}
\dot{\hat{x}} &= f(\hat{x}, u, 0) + L(y - C\hat{x}) \\
\dot{P} &= (\bar{A} - LC)P + P(\bar{A} - LC)^T + \bar{F}Q\bar{F}^T + LRL^T \\
L &= PC^TR^{-1} \\
P(t_0) &= E\{x(t_0)x^T(t_0)\}
\end{align*}
\]

\[
\bar{A} = \frac{\partial f}{\partial x}(0, \hat{x}, u, 0) = \frac{\partial f}{\partial x}(\hat{x}, u, 0) \\
\bar{F} = \frac{\partial f}{\partial v}(0, \hat{x}, u, 0) = \frac{\partial f}{\partial v}(\hat{x}, u, 0)
\]

- Little formal theory, but works very well as long as estimated state is close
- Very important for tracking problems (might operate far from equilibrium)
Example: Ducted Fan

Equations of motion

\[
\begin{align*}
    m\ddot{x} &= f_1 \cos \theta - f_2 \sin \theta - c_{d,x}(\theta, \dot{x}) \\
    m\ddot{y} &= f_1 \sin \theta + f_2 \cos \theta - mg - c_{d,y}(\theta, \dot{y}) \\
    J\ddot{\theta} &= rf_1 - mgl \sin \theta - c_{d,\theta}(\theta, \dot{\theta})
\end{align*}
\]

Estimator design: see dfan_kf.m, pvtol.m

Estimation:

- Given the \(xy\) position of the fan and the inputs \((f_1, f_2)\), determine the full state of the system:

\[x, y, \theta, \dot{x}, \dot{y}, \dot{\theta}\]
Variation #3: Parameter Estimation

Suppose dynamics depend on unknown parameter $\xi$

\[
\begin{align*}
\dot{x} &= A(\xi)x + B(\xi)u + Fv \\
y &= C(\xi)x + w
\end{align*}
\xi \in \mathbb{R}^p
\]

Rewrite dynamics using added state $\bar{\xi}$

\[
\begin{align*}
\dot{x} &= A(\xi)x + B(\xi)u + Fv \\
\dot{\xi} &= 0
\end{align*}
\]

Now use extended Kalman filter to estimate state and parameter:

\[
\frac{d}{dt} \begin{bmatrix} x \\ \xi \end{bmatrix} = \begin{bmatrix} A(\xi) & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ \xi \end{bmatrix} + \begin{bmatrix} B(\xi) \\ 0 \end{bmatrix} u + \begin{bmatrix} F \\ 0 \end{bmatrix} v
\]

\[
y = C(\xi)x + w
\]

\[
h(\begin{bmatrix} x \\ \xi \end{bmatrix}, w)
\]
Variation #4: Discrete Time (ala Wikipedia)

Discrete-time dynamical system:

\[ x_k = F x_{k-1} + B u_k + w_k \]
\[ z_k = H x_k + v_k \]

Kalman filter:

\[ \hat{x}_{k|k-1} = F_k \hat{x}_{k-1|k-1} + B_k u_k \]
\[ P_{k|k-1} = F_k P_{k-1|k-1} F_k^T + Q_k \]
\[ \tilde{y}_k = z_k - H_k \hat{x}_{k|k-1} \]
\[ S_k = H_k P_{k|k-1} H_k^T + R_k \]
\[ K_k = P_{k|k-1} H_k^T S_k^{-1} \]
\[ \hat{x}_{k|k} = \hat{x}_{k|k-1} + K_k \tilde{y}_k \]
\[ P_{k|k} = (I - K_k H_k) P_{k|k-1} \]

where \( x \in \mathbb{R}^n, u \in \mathbb{R}^m \)
\( w, v \) Gaussian white noise sequences

w/ covariance matrices \( Q \) and \( R \).

innovation or measurement residual
innovation (or residual) covariance
Kalman gain
updated state estimate
updated estimate covariance
Course Overview

Estimation and Kalman filtering
- Given measurement of process input and output, predict current state
- Use stochastic description of noise and disturbance processes
- Optimal (Kalman) estimator: minimize covariance of the error
- Many variants: nonlinear (EKF), parameter estimation, discrete time
- HW 4: design several estimators using (continuous time) Kalman filter

Next
- Discuss robust performance of feedback systems (following DFT)