Matrix Diagonalization and Systems of ODEs

Outline

- Why do we care of matrix diagonalization?
- What are eigenvalues and eigenvectors?
- How do we compute them?
- How do we use eigenvalues and eigenvectors to diagonalize a matrix?
- How do we solve systems of ODEs?
- How to infer stability information from the eigenvalues
Motivation to Diagonalization

\[ m_1 = m_2 = m \]
\[ k_1 = k_2 = k_3 = k \]
\[ u(t) = 0 \]

Figure 1: Mass spring system

\[ m \ddot{x}_1 = -2kx_1 + kx_2 - b \dot{x}_1 \]
\[ m \ddot{x}_2 = kx_1 - 2kx_2 - b \dot{x}_2 \]

WHAT IS THE SOLUTION \((x_1(t), x_2(t))\)?

**Hint:** yesterday you saw the solution of odes of the kind

\[ my + by + ky = 0 \]
Try to change the coordinates:

\[ z_1 = \frac{1}{2}(x_1 + x_2) \]
\[ z_2 = \frac{1}{2}(x_2 - x_1) \] (1)

then

\[ m\ddot{z}_1 = -kz_1 - b\dot{z}_1 \]
\[ m\ddot{z}_2 = -kz_2 - b\dot{z}_2 \]

which is now decoupled.

Then you solve the first one to find \( z_1(t) \) and the second one to find \( z_2(t) \), and inverting (1) you find

\[ x_1(t) = z_1(t) - z_2(t) \]
\[ x_2(t) = z_2(t) + z_1(t) \] (2)

and you have solved the problem.
In matrix notation we can write

\[ x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \quad z = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \]

and change of coordinates (2) becomes

\[ x = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} z \]

and the inverse change of coordinates (1) becomes

\[ z = \begin{pmatrix} 1/2 & 1/2 \\ -1/2 & 1/2 \end{pmatrix} x. \]

Let

\[ P = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \]

and

\[ p^{-1} = \begin{pmatrix} 1/2 & 1/2 \\ -1/2 & 1/2 \end{pmatrix}. \]

We can picture this in the following schematic:

\[ \begin{array}{c}
\text{ODEs in } x \\
\xrightarrow{P^{-1}} \\
\text{solution } x(t) \xleftarrow{P} \\
\text{ODEs: decoupled} \\
\xrightarrow{\text{solve for } z(t)} \\
\end{array} \]

Figure 2: Schematic for spring mass ODEs solution
In general for $x \in \mathbb{R}^n$ and $A \in \mathbb{R}^{n\times n}$ and the system of ODEs

$$\dot{x} = Ax$$

we want to find (if it exists) the change of coordinates represented by the matrix $P$ such that

$$P^{-1}AP = \Lambda, \quad \Lambda = \begin{pmatrix} \lambda_1 & 0 & \cdots & \cdots & 0 \\ 0 & \lambda_2 & 0 & \cdots & 0 \\ \vdots & \cdots & \ddots & \ddots & \vdots \\ \vdots & \cdots & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & \cdots & \lambda_n \end{pmatrix}$$

because if we change the coordinates in the $z$ variables where

$$z = P^{-1}x \quad \text{and} \quad x = Pz$$

we have

$$\dot{z} = P^{-1}\dot{x} = P^{-1}Ax = P^{-1}APz = \Lambda z$$

which means that we have $n$ decoupled dynamics which we can treat independently. In fact $\dot{z} = \Lambda z$ can be rewritten in scalar form as

$$\begin{align*}
\dot{z}_1 &= \lambda_1 z_1 \\
\dot{z}_2 &= \lambda_2 z_2 \\
&\vdots \\
\dot{z}_n &= \lambda_n z_n
\end{align*}$$

which are $n$ first order ODEs that we can solve independently as

$$z_i(t) = z_i(0)e^{\lambda_i t}$$

for all $i$, and then we can go back to the $x$ coordinates so to get the $x(t)$ solution as

$$x(t) = Pz(t).$$
This is one of the reasons why it is useful to find a change of coordinates $P$ that transforms matrix $A$ to its diagonal form $\Lambda$ (when it exists).
The process of finding $P$ and the diagonal matrix $\Lambda$ is called **diagonalization**. This process needs the computation of eigenvalues and eigenvectors of the matrix $A$. 
Eigenvectors and Eigenvalues

Let $A \in \mathbb{R}^{n \times n}$. A vector $v \in \mathbb{R}^n$ is said to be an eigenvector of $A$ with eigenvalue $\lambda$ if

$$Av = \lambda v$$

Figure 3: A matrix acts on its eigenvectors by scaling them.

How do we find the eigenvalues of $A$?

**Theorem:** $\lambda$ is an eigenvalue of $A$ if and only if

$$\text{det}(A - \lambda I) = 0$$

($I$ is the $n \times n$ identity matrix)

The expression $\text{det}(A - \lambda I)$ is a function of $\lambda$ of the form

$$\lambda^n + c_1\lambda^{n-1} + \ldots + c_{n-1}\lambda + c_n$$

which is called the **characteristic polynomial** of $A$. Then to find the eigenvalues of the matrix $A$ we need to find the roots of the characteristic polynomial.
EXAMPLE (step 1)

Let

\[ A = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix} \]

Compute the eigenvalues of A.
How do we find the eigenvalues of $A$?

Once you have found the eigenvalues $\lambda_i$ of the matrix $A$, you can find the corresponding eigenvector $v_i$ by solving the system of equations

$$(A - \lambda I)v_i = 0$$

for $v_i$.

EXAMPLE (step 2)

Let

$$A = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}$$

On the basis of the computed eigenvalues of $A$ (step 1) compute the corresponding eigenvectors of $A$. 
Diagonalization

Diagonalization theorem: If the eigenvalues of an \( n \times n \) matrix are real and distinct, then any set of corresponding eigenvectors \( \{v_1, \ldots, v_n\} \) form a matrix \( P = (v_1, \ldots, v_n) \) that is invertible and

\[
P^{-1}AP = \Lambda
\]

where

\[
\Lambda = \begin{pmatrix}
\lambda_1 & 0 & \cdots & 0 \\
0 & \lambda_2 & 0 & \cdots \\
\vdots & \ddots & \ddots & \vdots \\
0 & \cdots & \cdots & \lambda_n
\end{pmatrix}
\]

and \( \lambda_i \) is the eigenvalue with eigenvector \( v_i \).

Idea: assume for example that \( A \in \mathbb{R}^{2 \times 2} \) and \( v_1 \) and \( v_2 \) are eigenvectors of \( A \). Then

\[
P = (v_1, v_2)
\]

and

\[
P^{-1} = \begin{pmatrix}
w_1^T \\
w_2^T
\end{pmatrix}
\]

where \( w_1 \) and \( w_2 \) are vectors such that \( w_1^Tv_1 = 1, w_1^Tv_2 = 0, w_2^Tv_1 = 0 \) and \( w_2^Tv_2 = 1 \). Then

\[
P^{-1}AP = P^{-1}(Av_1, Av_2) = P^{-1}(\lambda_1v_1, \lambda_2v_2) = \begin{pmatrix}
\lambda_1w_1^Tv_1 & \lambda_2w_1^Tv_2 \\
\lambda_1w_2^Tv_1 & \lambda_2w_2^Tv_2
\end{pmatrix}
\]

\[
= \begin{pmatrix}
\lambda_1 & 0 \\
0 & \lambda_2
\end{pmatrix}
\]
EXAMPLE (step3)
given the matrix A of steps 1 and 2, compute P and verify that
$P^{-1}AP = \Lambda$. 

11
Solution of Systems of ODEs

Given the linear dynamical system
\[ \dot{x} = Ax \]
let \( \{v_1, \ldots, v_n\} \) be a basis of eigenvectors with eigenvalues \( \{\lambda_1, \ldots, \lambda_n\} \).
Let \( P = (v_1, \ldots, v_n) \), and consider the change of coordinates
\[ z = P^{-1}x \]
then the dynamics in the new coordinates \( z \) becomes
\[ \dot{z} = P^{-1}\dot{x} = P^{-1}Ax = P^{-1}APz \]
then by virtue of the diagonalization theorem \( P^{-1}AP = \Lambda \), so that
\[ \dot{z} = \Lambda z \]
that is
\[ \dot{z}_i = \lambda_i z_i \quad \text{for all } i \]
which have solutions
\[ z_i(t) = z_i(0)e^{\lambda_i t} \quad \text{for all } i \]
so that
\[ z(t) = \begin{pmatrix} z_1(t) \\ z_2(t) \\ \vdots \\ z_n(t) \end{pmatrix} \]
and
\[ x(t) = Pz(t) \]
EXAMPLE
consider $\dot{x} = Ax$ with

$$A = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}$$

find $x(t)$. 
Stability of Systems of ODEs from Eigenvalues

Given the system $\dot{x} = Ax$, once we have the solution $x(t)$, we will say that the system is **unstable** if $\|x(t)\|$ becomes arbitrarily far from the equilibrium point, $x=0$, as time increases. Since $P$ is not depending on time, we expect that if $\|z(t)\|$ is becoming arbitrarily big as time increases, then also $\|x(t)\|$ will, and if $\|z(t)\|$ is staying close to 0 for any time, also $\|x(t)\|$ is. Then to check if the system is unstable we can check the behavior of $\|z(t)\|$ in time.

![Graph]

**Figure 4:** Stability property of the system of ODEs depending on the eigenvalues of $A$. 
Useful Matlab Commands

1. $[V,D]=\text{eig}(A)$ gives the eigenvalues of $A$ in the matrix $D$ and the eigenvectors of $A$ in the matrix $V$.

Useful References
