CDS 201 Qualifying Examination, January 2002.

Answer all parts of the following questions.

1. Consider the following matrix

\[ A = \begin{pmatrix} -1 & 2 & 3 \\ 0 & -4 & 5 \\ 0 & 0 & -4 \end{pmatrix}. \]

Find its Jordan canonical form.

2. Consider the following ordinary differential equation:

\[ \dot{x} = Ax, \]

where \( A \) is defined as in problem 1. The positive orbit of \( x_0 \) under the flow generated by this differential equation is given by

\[ O(x_0) = \{ x \mid x = e^{At} x_0, \ t \geq 0 \}. \]

(a) Compute the exponential \( e^{tA} \) in terms of the matrix \( P \) that brings \( A \) into Jordan canonical form.

(b) For which values of \( x_0 \) are the positive orbits bounded? compact?

3. Let \( M^{n \times n} \) denote the set of \( n \times n \) matrices and define the map \( f : M^{n \times n} \to M^{n \times n} \) by \( f(A) = AA^T \).

(a) Show that \( f \) is a smooth mapping

(b) Compute the derivative of \( f \) at the identity \( I \in M^{n \times n} \).

(c) Is the derivative surjective at any points?

(d) Expand \( f \) in a Taylor series about the identity.

4. True or false? (Justify your answer.) If \( \epsilon > 0 \) is sufficiently small, and \( |a|, |b| < \epsilon \), then the following system of equations has a solution \((x, y)\) near \((0, 0)\).

\[ a = \sin(x + y) - \cos(x - y) + 1, \]

\[ b = \sin(x - y) + \cos(x + y) - 1. \]