CDS 201 Qualifying Examination, April 2003.

Answer all parts of the following two questions.

1. Let $\alpha$ and $\lambda$ be real numbers.

(a) For $\alpha \neq 0$, explain in what sense the matrix

$$ P(\alpha) = \begin{pmatrix} 0 & 0 & 1/\alpha^2 \\ 0 & 1/\alpha & 0 \\ 1 & 0 & 0 \end{pmatrix} $$

transforms the matrix

$$ A(\alpha, \lambda) = \begin{pmatrix} \lambda & 0 & 0 \\ \alpha & \lambda & 0 \\ 0 & \alpha & \lambda \end{pmatrix} $$

into Jordan canonical form. What is this canonical form? Discuss the case $\alpha = 0$.

(b) Compute $e^{A(\alpha, \lambda)t}$

(c) Find a bound for the matrix norm of $e^{A(\alpha, \lambda)t}$ in terms of $\alpha, \lambda$ and $t$.

(d) For what values of $x_0 \in \mathbb{R}^3, \alpha, \lambda$, is the set

$$ O(x_0, \alpha, \lambda) \equiv \left\{ x \in \mathbb{R}^3 \mid x = e^{A(\alpha, \lambda)t}x_0, t \geq 0 \right\}, $$

a bounded set? A closed set? A compact set?

2. Let $\epsilon > 0$ be a positive constant and let $C$ be a real number. Let $E$ be the Banach space of continuous functions $f : [0, \epsilon] \rightarrow \mathbb{R}$ with the norm $\|f\| = \sup_{x \in [0, \epsilon]} |f(x)|$. Let

$$ B_r = \{ g \in E \mid \|g - 1\| \leq r \} $$

be the ball of radius $r > 0$ centered at the constant function 1 with radius $r$. Consider the mapping $T : E \rightarrow E$ defined by

$$ (Tf)(x) = 1 + C \int_0^x |f(s)|^2 ds, $$

(a) Find conditions on $r, \epsilon, C$ such that $T$ maps $B_r$ to itself.

(b) Find conditions on $r, \epsilon, C$ such that $T$ satisfies an estimate of the form

$$ \|T(f_1) - T(f_2)\| \leq K\|f_1 - f_2\| $$

for a constant $K$ and all $f_1, f_2 \in B_r$.

(c) Find conditions on $r, \epsilon, C$ such that $T$ is a contraction mapping of $B_r$ to itself.

(d) Explain why for such $r, \epsilon, C$ the map $T$ must have a unique fixed point $f_0$.

(e) Find an explicit formula for this fixed point $f_0$. 