CDS140a Candidacy Exam

Problem 1: Simple pendulum with driven support

Consider a point mass \( m \) connected by a massless rod of length \( l \) to a vertically driven support, whose time-dependent position is given by

\[
\zeta (t) = \zeta_0 + A \sin \omega t.
\]

Assume that gravity is the only external force.

(a) Find the Lagrangian for this system using \( \theta \), the angle the pendulum makes with the vertical, as the generalized coordinate.

(b) Derive the Euler-Lagrange equation of motion.

(c) Introducing an extra variable to account for the time-dependence, rewrite your answer as a nonlinear system of the form

\[
\frac{d}{dt} \vec{x} = f (\vec{x}).
\]

(d) Linearize your equations about the periodic orbits corresponding to \( \dot{\theta} = 0 \).

Problem 2: Planar systems

(a) Consider a Hamiltonian mechanical system (one degree of freedom) with potential energy function

\[
V (x) = V_0 \exp (-x^2),
\]

where \( V_0 \) is a real positive constant. Assume the usual kinetic energy term, and let the particle have unit mass. Find the stable and unstable manifolds of the saddle at the origin \( x = 0 \).

(b) Draw the phase portrait for a simple pendulum, and identify all of the possible \( \omega \)-limit sets.

(c) What are the possible \( \omega \)-limit sets for a damped pendulum?