CDS 101: Lecture 4-1
Reachability and State Space Feedback

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Goals:
- Define reachability of a control system
- Give tests for reachability of linear systems and apply to examples
- Describe the design of state feedback controllers for linear systems

Reading:
Åström and Murray, *Feedback Systems*, Ch 6

Review from last week

Properties of Linear Systems:
- Linearity with respect to inputs and initial conditions
- Stability characterized by eigenvalues
- Response described by convolution integral
- Many applications and tools
- Characterizes a nonlinear system around an equilibrium point
- *No system is linear for all input amplitudes, but linearization is a good enough model for many systems*
Control Overview

- Trajectory Generation
- Controller
- Process
  \[ u \]

**Design controller so that**

1. **System is stable**
2. **Performance:**
   - Keep close to equilibrium despite disturbances (disturbance rejection)
   - Move system to desired state (track reference)
3. **Robustness to modeling errors**

\[ \lambda \approx 5.8 \text{ rad/sec} \]

\( \times 2 \text{ in } 120 \text{ msec} \)

First Questions...

1. **Can the input** \( u \) **affect the dynamics?**
   
   \[ \dot{x}_1 = x_1 + u \]
   
   \( \text{e.g.} \)
   
   \[ \dot{x}_2 = x_2 \]
   
   \( \Rightarrow \) Can’t change \( x_2 \)

   Equivalent to asking whether there is a \( u \) that allows us to reach any point in the state-space

   \( \Rightarrow \) Reachability (today), depends on \( A, B \)

   \( \Rightarrow \) Related to the design of state feedback

2. **Does the measurement** \( y \) **contain enough information about the system?**

   \[ \dot{x}_1 = x_1 \quad y = x_1 \]
   
   \( \text{e.g.} \)
   
   \[ \dot{x}_2 = x_2 \]
   
   \( \Rightarrow \) Can’t measure \( x_2 \)

   \( \Rightarrow \) Observability (next week), depends on \( A, C \)

   \( \Rightarrow \) Related to the design of observers to estimate state from measurement
**Control Design Concepts**

**System description:** single input, single output system (MIMO also OK)

\[ \begin{align*}
\dot{x} &= f(x, u) \quad x \in \mathbb{R}^n, \quad x(0) \text{ given} \\
y &= h(x, u) \quad u \in \mathbb{R}, \quad y \in \mathbb{R}
\end{align*} \]

**Stability:** stabilize the system around an equilibrium point

- Given equilibrium point \( x_e \in \mathbb{R}^n \), find control “law” \( u = \alpha(x) \) such that \( \lim_{t \to \infty} x(t) = x_e \forall x(0) \in \mathbb{R}^n \)

**Reachability:** steer the system between two points

- Given \( x_0, x_f \in \mathbb{R}^n \), find an input \( u(t) \) such that \( \dot{x} = f(x, u(t)) \) takes \( x(t_0) = x_0 \to x(T) = x_f \)

**Tracking:** track a given output trajectory

- Given \( y_d(t) \), find \( u = \alpha(x, t) \) such that \( \lim_{t \to \infty} (y(t) - y_d(t)) = 0 \), \( \forall x(0) \in \mathbb{R}^n \)

**Reachability of Input/Output Systems**

\[ \begin{align*}
\dot{x} &= f(x, u) \quad x \in \mathbb{R}^n, \quad x(0) \text{ given} \\
y &= h(x, u) \quad u \in \mathbb{R}, \quad y \in \mathbb{R}
\end{align*} \]

**Defn** An input/output system is **reachable** if for any \( x_0, x_f \in \mathbb{R}^n \) and any time \( T > 0 \) there exists an input \( u_{[0,T]} \in \mathbb{R}^n \) such that the solution of the dynamics starting from \( x(0) = x_0 \) and applying input \( u(t) \) gives \( x(T) = x_f \).

**Remarks**

- In the definition, \( x_0 \) and \( x_f \) do not have to be equilibrium points \( \Rightarrow \) we don’t necessarily stay at \( x_f \) after time \( T \).
- Reachability is defined in terms of states \( \Rightarrow \) doesn’t depend on output
- For **linear systems**, can characterize reachability by looking at the general solution:

\[ \begin{align*}
\dot{x} &= Ax + Bu \\
y &= Cx + Du \\
x(T) &= e^{AT}x_0 + \int_{\tau=0}^{T} e^{A(T-\tau)}Bu(\tau)d\tau
\end{align*} \]
Tests for Reachability

\[ \dot{x} = Ax + Bu \quad x \in \mathbb{R}^n, \ x(0) \text{ given} \]
\[ y = Cx + Du \quad u \in \mathbb{R}, \ y \in \mathbb{R} \]
\[ x(T) = e^{AT}x_0 + \int_{\tau=0}^{T} e^{A(T-\tau)}Bu(\tau)d\tau \]

**Thm** A linear system is reachable if and only if the \( n \times n \) reachability matrix
\[
\begin{bmatrix}
B & AB & A^2B & \cdots & A^{n-1}B
\end{bmatrix}
\]
is full rank.

**Remarks**
- Very simple test to apply. In MATLAB, use `ctrb(A,B)` and check rank w/ `det()`
- If this test is satisfied, we say "the pair (A,B) is reachable"
- Some insight into the proof can be seen by expanding the matrix exponential
  \[
e^{A(T-\tau)}B = \left(I + A(T-\tau) + \frac{1}{2}A^2(T-\tau)^2 + \cdots + \frac{1}{(n-1)!}A^{n-1}(T-\tau)^{n-1} + \cdots\right)B
  = B + AB(T-\tau) + \frac{1}{2}A^2B(T-\tau)^2 + \cdots + \frac{1}{(n-1)!}A^{n-1}B(T-\tau)^{n-1} + \cdots
\]
- Test does not give a measure of how much control effort is required
- Other tests for reachability also exist

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**Example #1: Linearized pendulum on a cart**

Question: can we locally control the position of the cart by proper choice of input?

Approach: look at the linearization around the upright position (good approximation to the full dynamics if \( \theta \) remains small)

\[
\frac{d}{dt} \begin{bmatrix} p \\ \theta \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & \frac{m_g \theta}{J_m - m \theta^2} & -\frac{J_m}{M J_m - m \theta^2} & \frac{m_g \theta}{J_m} \\ 0 & \frac{m \theta}{M J_m - m \theta^2} & -\frac{J_m}{M J_m - m \theta^2} & \frac{m \theta}{J_m} \end{bmatrix} \begin{bmatrix} p \\ \theta \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \frac{J_m \dot{\theta}}{M J_m - m \theta^2} \\ \frac{J_m \dot{\theta}}{M J_m - m \theta^2} \end{bmatrix} u
\]
\[ y = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix} x, \]
Example #1, con’t: Linearized pendulum on a cart

\[
\begin{align*}
\dot{\theta} &= \frac{m^2 g}{\mu} - \frac{c \theta}{\mu} \\
\dot{\omega} &= -\frac{M g m^2}{\mu^2} - \frac{\omega}{\mu}
\end{align*}
\]

\[
\mu = M_1 J_1 - m^2 \mu^2
\]

Reachability matrix:

\[
W_r = \begin{bmatrix}
0 & \frac{J_1}{\mu} & 0 \\
0 & \frac{i m}{\mu} & 0 \\
\frac{i m}{\mu} & 0 & \frac{g l^2 m^3}{\mu^2} & \frac{g l^2 m^3 (m+M)}{\mu^2} & 0 \\
B & \frac{i m}{\mu} & 0 & \frac{g l^2 m^3 (m+M)}{\mu^2} & A^2 B \\
& & & & A^3 B
\end{bmatrix}
\]

- Full rank as long as constants are such that columns 1 and 3 are not multiples of each other
- Reachable as long as \(\det(W_r) = \frac{g^2 c^4 M^4}{\mu^4} \neq 0\)
- Can “steer” linearization between points by proper choice of input

Control Design Concepts

System description: single input, single output system (MIMO also OK)

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y &= h(x, u) & u \in \mathbb{R}, y \in \mathbb{R}
\end{align*}
\]

Stability: stabilize the system around an equilibrium point
- Given equilibrium point \(x_e \in \mathbb{R}^n\), find control “law” \(u = \alpha(x)\) such that \(\lim_{t \to \infty} x(t) = x_e, \forall x(0) \in \mathbb{R}^n\)

Reachability: steer the system between two points
- Given \(x_0, x_f \in \mathbb{R}^n\), find an input \(u(t)\) such that \(\dot{x} = f(x, u(t))\) takes \(x(t_0) = x_0 \to x(T) = x_f\)

Tracking: track a given output trajectory
- Given \(y_d(t)\), find \(u = \alpha(x, t)\) such that \(\lim_{t \to \infty} (y(t) - y_d(t)) = 0, \forall x(0) \in \mathbb{R}^n\)
State space controller design for linear systems

\[ \dot{x} = Ax + Bu \quad x \in \mathbb{R}^n, \quad x(0) \text{ given} \]
\[ y = Cx + Du \quad u \in \mathbb{R}, \quad y \in \mathbb{R} \]
\[ x(T) = e^{AT}x_0 + \int_{\tau=0}^{T} e^{A(T-\tau)}Bu(\tau)d\tau \]

**Goal:** find a linear control law \( u = -Kx \) such that the closed loop system
\[ \dot{x} = Ax - BKx = (A - BK)x \]
is stable at \( x_e = 0 \).

**Remarks**
- Stability based on eigenvalues \( \Rightarrow \) use \( K \) to make eigenvalues of \( (A - BK) \) stable
- Can also link eigenvalues to performance (e.g., initial condition response)
- Question: when can we place the eigenvalues any place that we want?

**Theorem** The eigenvalues of \( (A - BK) \) can be set to arbitrary values if and only if the pair \((A, B)\) is reachable.

MATLAB: \( K = \text{place}(A, B, \text{eigs}) \)

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**Example #2: Predator prey**

**Natural dynamics**
\[ \frac{dH}{dt} = rH \left( 1 - \frac{H}{k_c} \right) - \frac{aHL}{c + H}, \quad H \geq 0 \]
\[ \frac{dL}{dt} = b - \frac{aHL}{c + H} - dL, \quad L \geq 0 \]

**Controlled dynamics: modulate food supply**
\[ \frac{dH}{dt} = (r + u)H \left( 1 - \frac{H}{k_c} \right) - \frac{aHL}{c + H} \]
\[ \frac{dL}{dt} = b - \frac{aHL}{c + H} - dL, \]

**Q1:** can we move from some initial population of lynxes and hares to a specified one in time \( T \) by modulation of the food supply?

**Q2:** can we stabilize the population around the desired equilibrium point

**Approach:** try to answer this question locally, around the natural equilibrium point
Example #2: Problem setup

Equilibrium point calculation

\[ \frac{dH}{dt} = (r + u)H \left( 1 - \frac{H}{k_c} \right) - \frac{aHL}{c + H} \]
\[ \frac{dL}{dt} = b \frac{aHL}{c + H} - dL, \]

- \( x_e = (20.5, 29.5) \), \( u_e = 0 \)

Linearization

- Compute linearization around equil. point, \( x_e \):
  \[ A = \frac{\partial f}{\partial x}|_{x_e,u_e} \quad B = \frac{\partial f}{\partial u}|_{x_e,u_e} \]

- Redefine local variables: \( z = x - x_e \), \( v = u - u_e \)

Control design:

- Choose \( \lambda = -1, -2 \)
- \( K = \text{place}(A, B, [-1; -2]) \)

Modify NL dynamics to include control

\[ \frac{dH}{dt} = (r + u)H \left( 1 - \frac{H}{k_c} \right) - \frac{aHL}{c + H} \]
\[ \frac{dL}{dt} = b \frac{aHL}{c + H} - dL, \]

% Compute the equil point
% predprey.m contains dynamics
\texttt{f = inline('predprey(0,x)');}
\texttt{xeq = fsolve(f, [20, 30]);}

% Compute linearization
\texttt{A = [...];}
\texttt{B = [H0*(1 - H0/K); 0];}
\texttt{p = [-1; -2];}
\texttt{K = place(A,B,p)}

Example #2: Stabilization via eigenvalue assignment

\[ \frac{d}{dt} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} r - \frac{2H_0c}{k} - \frac{aL_0}{c + H_0} + \frac{aL_0H_0}{(c + H_0)^2} - \frac{aH_0}{c + H_0} \\ baL_0 \left( \frac{1}{c + H_0} - \frac{H_0}{(c + H_0)^2} \right) ba - \frac{H_0}{c + H_0} - d \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} + \begin{bmatrix} H_0 \left( 1 - \frac{H_0}{k} \right) \\ 0 \end{bmatrix} v \]

Control design:

- \( v = -Kz + k_v \)
- \( u = u_e - K(x - x_e) + k_v(r - y_e) \)

Place poles at stable values

- Choose \( \lambda = -1, -2 \)
- \( K = \text{place}(A, B, [-1; -2]) \)

Modify NL dynamics to include control

\[ \frac{dH}{dt} = (r + u)H \left( 1 - \frac{H}{k_c} \right) - \frac{aHL}{c + H} \]
\[ \frac{dL}{dt} = b \frac{aHL}{c + H} - dL, \]

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{example2_plot.png}
\caption{Stable}
\end{figure}
Remarks:
1. In practice, don’t always have access to full state \( \rightarrow \) estimate state from measurement \( y \) (next week)
2. What to pick for eigenvalues?
   - For each eigenvalue \( \lambda = \sigma_i + j\omega_i \), get contribution of the form \( y_i(t) = e^{\sigma_i t}(a \sin(\omega_i t) + b \cos(\omega_i t)) \)
   - Faster response will require more control effort
   - Optimal control: LQR (in text, CDS 110b)
3. How to obtain desired tracking response so that \( y_{ss} = r \) for some reference \( r \)?
   - Choose \( u = -Kx + kr \)
   - Steady state (if \( D=0 \)): \( y = Cx = C(A - BK)^{-1}Br \Rightarrow k_r = \text{inv}[C(A - BK)^{-1}B] \)

Reachable Canonical Form

If the system is reachable, then there exists a transformation \( z = Tx \) such that:

\[
\dot{z} = \begin{bmatrix}
-a_1 & -a_2 & -a_3 & \cdots & -a_n \\
1 & 0 & 0 & \cdots & 0 \\
0 & 1 & 0 & \cdots & 0 \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
0 & \cdots & 1 & 0
\end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ u \\ \vdots \\ 0 \end{bmatrix}
\]

(Check \([B AB A^2B \ldots A^{n-1}B] \ldots \) is this system reachable?)

Characteristic polynomial is:

\[
\lambda(s) = s^n + a_1s^{n-1} + \cdots + a_{n-1}s + a_n
\]

Choose state feedback:

\[
u = -Kz = -\begin{bmatrix} k_1 & k_2 & \cdots & k_n \end{bmatrix}z
\]

Then closed-loop system is:

\[
\dot{z} = \begin{bmatrix}
-a_1 - k_1 & -a_2 - k_2 & -a_3 - k_3 & \cdots & -a_n - k_n \\
1 & 0 & 0 & \cdots & 0 \\
0 & 1 & 0 & \cdots & 0 \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
0 & \cdots & 1 & 0
\end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}
\]
Summary: Reachability and State Space Feedback

\[
\begin{align*}
\dot{x} &= Ax + Bu \\
y &= Cx + Du
\end{align*}
\]

Key concepts
- Reachability: find \( u \) s.t. \( x_0 \rightarrow x_f \)
- Reachability rank test for linear systems
- State feedback to assign eigenvalues

\[
u = -Kx + k_p r
\]