Lecture 6

10. Summary

In Sections 1 – 9, the following estimation methods were presented:

1) The Gradient Estimator (GE)
2) The Normalized Gradient Estimator (NGE)
3) The Least-Squares Estimator (LSE)
4) The Normalized Least-Squares Estimator with Exponential Forgetting (NLSEEF)

Their properties and related equations are summarized below.

- **GE** is simple but has slow convergence.
- **LSE** is robust to noise but cannot estimate time-varying parameters.
- **NLSEEF** has the ability to track time-varying parameters but there is a possibility of gain windup in the absence of persistent excitation.
  - The gain windup problem can be overcome by employing the bounded gain forgetting technique.
- **Truth model** is static and linear in parameters.

\[
y(t) = \theta^T \Phi(x(t))
\]

- **Prediction model** is:

\[
\hat{y}(t) = \hat{\theta}^T (t) \Phi(x(t))
\]

- Prediction error is:

\[
e_y = \hat{y} - y
\]

- On-line parameter estimation law is the same in all the methods.

\[
\begin{align*}
\dot{\hat{\theta}} &= -\Gamma \Phi e_y \\
\dot{\hat{\theta}} &= -\Gamma \frac{\Phi}{(1 + \Phi^T P \Phi)} e_y \overset{\text{Normalized Estimation}}{=} n_{\hat{\theta}}
\end{align*}
\]

- GE and NGE use constant symmetric positive definite gain matrix \( \Gamma \).
• Estimation gain matrix update in LSE is:
\[
\begin{align*}
\dot{\Gamma} &= -\Gamma \Phi \Phi^T \Gamma \\
\dot{\Gamma} &= -\Gamma \frac{\Phi \Phi^T}{n_\Phi} \Gamma \quad \text{Normalized Estimation}
\end{align*}
\]

• Estimation gain matrix update in LSEEF is:
\[
\begin{align*}
\dot{\Gamma} &= \lambda(t) \Gamma - \Gamma \Phi \Phi^T \Gamma \\
\dot{\Gamma} &= \lambda(t) \frac{\Phi \Phi^T}{n_\Phi} \Gamma \quad \text{Normalized Estimation}
\end{align*}
\]

• In LSEBGF, the time-varying factor \( \lambda(t) \) is chosen to yield bounded gain forgetting process:
\[
\lambda(t) = \lambda_0 \left( 1 - \frac{\|\Gamma(t)\|}{k_0} \right)
\]

• Persistent Excitation (PE) is essential for good parameter estimation but is not required for good output prediction.
• In order to have good estimation performance, the following implementation issues must be considered:
  o Generating linear-in-parameters model for on-line estimation purposes
  o Choosing initial parameters and initial gain matrix
  o Specifying forgetting rate and gain bound
  o Enforcing persistency of excitation through system input

11. System Identification using Dynamic Linear in Parameters Models

In Section 3, using Linear Time-Invariant (LTI) systems, the concept of a dynamic linear in parameters model was introduced. In this section, we generalize this concept and design on-line system identification methods for the systems whose dynamics linearly depends on the unknown parameters but is nonlinear in the system state and the control input vectors. Towards that end, consider the nonlinear scalar system represented by:
\[
\dot{x} = f(x,u) \tag{11.1}
\]
where \( x \in \mathbb{R}^n \) is the state vector and \( u \in \mathbb{R}^m \) is the vector of external inputs. In this case, the full state vector \( x \) is assumed to be available for measurement.
In most applications, the vector field \( f(x,u) \) is partially known. The known part of \( f \), usually referred to as the \emph{nominal} model, is derived by analytical methods using first principles or by off-line identification methods. Therefore, it is assumed that \( f \) can be decomposed as:

\[
f(x,u) = f_0(x,u) + F(x,u)
\]  \hspace{1cm} (11.2)

where \( F_0(x,u) \in \mathbb{R}^{n \times 1} \) represents the known system dynamics, and \( F(x,u) \in \mathbb{R}^{n \times 1} \) denotes the unknown difference between the system dynamics and its nominal part. Furthermore, it is assumed that the unknown portion of the system dynamics can be written as a linear combination of known basis functions \( \phi_i(x,u) \) and unknown constant coefficients:

\[
F(x,u) = \Theta^T \Phi(x,u)
\]  \hspace{1cm} (11.3)

where \( \Phi(x,u) = (\phi_1 \ldots \phi_N)^T \in \mathbb{R}^{N \times 1} \) represents the \emph{known regressor vector}, while \( \Theta \in \mathbb{R}^{N \times n} \) denotes the matrix of \emph{unknown constant parameters}.

Using (11.2) and (11.3), the system (11.1) can be represented as the \emph{dynamic linear in parameters model}:

\[
\dot{x} = f_0(x,u) + \Theta^T \Phi(x,u)
\]  \hspace{1cm} (11.4)

\textbf{Example 11.1}

If \( f(x,u) = Ax + Bu \) and \( f_0 = 0_{n \times 1} \) then

\[
\dot{x} = Ax + Bu = \begin{pmatrix} A & B \end{pmatrix} \begin{pmatrix} x \\ u \end{pmatrix} = \Theta^T \Phi(x,u)
\]  \hspace{1cm} (11.5)

Based on (11.4), the system state \emph{estimator / predictor dynamics} is introduced as:

\[
\dot{x} = A_{\text{ref}} (\hat{x} - x) + f_0(x,u) + \hat{\Theta}^T \Phi(x,u)
\]  \hspace{1cm} (11.6)

where \( A_{\text{ref}} \) is any Hurwitz matrix and \( \hat{\Theta} \) is the matrix of on-line estimated parameters. Let

\[
e(t) = \hat{x}(t) - x(t)
\]  \hspace{1cm} (11.7)
denote the prediction error at time $t$. Subtracting (11.4) from (11.6), the prediction error dynamics can be written as:

$$
\dot{e} = A_{ref} e + (\hat{\Theta} - \Theta) \Phi(x,u) = A_{ref} e + \Delta \Theta^T \Phi(x,u)
$$

(11.8)

In (11.8), $\Delta \Theta = \hat{\Theta} - \Theta$ denotes the parameter estimation error matrix.

**Assumption 11.1**

The control input $u(t)$ in (11.4) is bounded, piecewise continuous and defined such that the state of the system $x(t)$ is bounded and evolves on a compact domain $X \subset \mathbb{R}^n$, that is: $u, x \in L_{\infty}$. In addition, it is assumed that given the chosen control input $u(t)$, the nominal dynamics $f_0(x,u(t))$ and the regressor vector $\Phi(x,u(t))$ are Lipschitz in $x$.

Let $Q = Q^T > 0$. Since $A_{ref}$ is Hurwitz, the algebraic Lyapunov equation

$$
P A_{ref}^T + A_{ref}^T P = -Q
$$

(11.9)

has the unique positive-definite symmetric matrix solution $P$. Using the latter, consider the following Lyapunov function candidate:

$$
V(e, \Delta \Theta) = e^T P e + \text{trace}(\Delta \Theta^T \Gamma^{-1} \Delta \Theta)
$$

(11.10)

where $\Gamma = \Gamma^T > 0$ is a symmetric and positive definite matrix, while

$$
\text{trace}(A) = \sum_i a_{ii}
$$

(11.11)

denotes the trace of a square matrix $A$, (i.e., the sum of its diagonal elements). We will use the well-known trace identity, which is valid for any 2 co-dimensional vectors, $v$ and $w$:

$$
v^T w = \text{trace}(wv^T) = \text{trace}(vw^T)
$$

(11.12)

Differentiating (11.10) along the trajectories of (11.8), yields:

$$
\dot{V}(e, \Delta \Theta) = \dot{e}^T P e + e^T P \dot{e} + 2 \text{trace}(\Delta \Theta^T \Gamma^{-1} \hat{\Theta})
$$

(11.13)
Substituting (11.8) into (11.13), regrouping terms, and using (11.9), gives:

\[
\dot{V}(e, \Delta \Theta) = \left( A_{ref} e + \Delta \Theta^T \Phi \right)^T P e + e^T P \left( A_{ref} e + \Delta \Theta^T \Phi \right) + 2 \text{trace} \left( \Delta \Theta^T \Gamma^{-1} \dot{\Theta} \right) \\
= e^T \left( P A_{ref} + A_{ref}^T e \right) e + 2 e^T P \Delta \Theta^T \Phi + 2 \text{trace} \left( \Delta \Theta^T \Gamma^{-1} \dot{\Theta} \right) \\
= -e^T Q e + 2 e^T P \Delta \Theta^T \Phi + 2 \text{trace} \left( \Delta \Theta^T \Gamma^{-1} \dot{\Theta} \right)
\]  
(11.14)

Using the trace identity (11.12), the 2\textsuperscript{nd} term in (11.14) can be written as:

\[
e^T P \Delta \Theta^T \Phi = \text{trace} \left( \Delta \Theta^T \Phi e^T P \right)
\]
(11.15)

Using (11.15), the time derivative in (11.14) becomes:

\[
\dot{V}(e, \Delta \Theta) = -e^T Q e + 2 \text{trace} \left( \Delta \Theta^T \left( \Gamma^{-1} \dot{\Theta} + \Phi e^T P \right) \right)
\]  
(11.16)

The \textit{parameter estimation laws} are chosen to make the time derivative in (11.16) negative semi-definite:

\[
\dot{\Theta} = -\Gamma \Phi e^T P
\]
(11.17)

Indeed, using (11.17) yields:

\[
\dot{V}(e, \Delta \Theta) = -e^T Q e \leq 0
\]  
(11.18)

Since \( V \geq 0 \) and \( \dot{V} \leq 0 \), the function \( V \) tends to a finite limit, the prediction error vector \( e(t) \) and the parameter estimation error matrix \( \Delta \Theta(t) \) are uniformly bounded in time and, consequently the matrix of estimated parameters \( \hat{\Theta}(t) \) is also bounded. Moreover, integrating (11.18), it is easy to see that \( e \in L_2 \).

Differentiating (11.18) along the trajectories of the system (11.8), gives:

\[
\ddot{V}(e, \Delta \Theta) = -2 e^T Q \dot{e} = -2 e^T Q \left( A_{ref} e + \Delta \Theta^T \Phi(x,u) \right)
\]
(11.19)

Since \( x, u \in L_2 \) and \( \Phi(x,u) \) is Lipschitz in \( x \), then the regressor vector is bounded: \( \Phi \in L_2 \). Using (11.17) and the fact that \( e \in L_2 \), immediately implies that \( \dot{\Theta} \in L_2 \).
Furthermore, from (11.19) it follows that the 2\textsuperscript{nd} derivative of $V$ is bounded: $\dot{V} \in L_\infty$. Consequently, $\dot{V}$ is uniformly continuous function of time. Also, it was proven that $V$ has a finite limit. Thus, using Barbalat Lemma implies that the function $\dot{V}$ in (11.18) asymptotically tends to zero, as $t \to \infty$. The latter is equivalent to:

$$\lim_{t \to \infty} \|e(t)\| = 0$$

(11.20)

that is the state $\dot{x}(t)$ of the predictor model (11.6) asymptotically converges to the state $x(t)$ of the original system (11.4).

Note that the parameter estimation law (11.17) has the form of a gradient-based estimator. Therefore, Persistency of Excitation (PE) is needed in order for the estimation error asymptotically converge to zero.

The properties of the on-line estimation law (11.17) for the dynamic linear in parameters model (11.4) are summarized below.

**Theorem 11.1**

Given the dynamic linear in parameters model (11.4)

$$\dot{x} = f_0(x,u) + \Theta^T \Phi(x,u)$$

suppose that $x, u \in L_\infty$ and that the functions $\{f_0(x,u), \Phi(x,u)\}$ are Lipschitz in $x$. Then using the state predictor dynamics (11.6)

$$\dot{x} = A_{ref}(\hat{x} - x) + f_0(x,u) + \hat{\Theta}^T \Phi(x,u)$$

along with the parameter estimation law (11.17)

$$\dot{\hat{\Theta}} = -\Gamma \Phi(x,u)e^T P$$

results in:

- $e, \dot{\hat{\Theta}} \in L_2$
- $e, \hat{\Theta} \in L_\infty$
- $\lim_{t \to \infty} \|\dot{\hat{x}}(t) - x(t)\| = 0$
If, in addition, the regressor vector \( \Phi \) is Persistently Exciting (PE) then the estimated parameters asymptotically converge to their true unknown constant values.

- \( \lim_{t \to \infty} \| \hat{\Theta}(t) - \Theta \| = 0 \)

**Example 11.2**

Consider again the uncertain LTI system (11.5) from Example 1.1.

\[
\dot{x} = A x + Bu = \begin{pmatrix} A & B \end{pmatrix} \begin{pmatrix} x \\ u \end{pmatrix} = \Theta^T \Phi(x,u)
\]

Suppose that \( A \) is Hurwitz. Then given bounded control input \( u \), the state of the system \( x \) will stay bounded. Thus, all the sufficient conditions of the Theorem 1.1 take place. In this case, using (11.6) the predictor dynamics is:

\[
\dot{x} = A_{ref} (\hat{x} - x) + \hat{A} x + \hat{B} u
\]

and the parameter estimation laws are:

\[
\begin{pmatrix} \hat{A}^T \\ \hat{B}^T \end{pmatrix} = -\Gamma \begin{pmatrix} x \\ u \end{pmatrix} e^T P
\]

or, equivalently:

\[
\begin{pmatrix} \hat{A} \\ \hat{B} \end{pmatrix} = -P e \begin{pmatrix} x^T \\ u^T \end{pmatrix} \Gamma
\]

Partition the estimation gain matrix:

\[
\Gamma = \begin{pmatrix} \Gamma_x & 0_{m \times m} \\ 0_{m \times n} & \Gamma_u \end{pmatrix}
\]

where \( \Gamma_x \in R^{n \times n} \) and \( \Gamma_u \in R^{m \times m} \) are the estimation gain matrices for \( \hat{A} \) and \( \hat{B} \), respectively. Then the parameter estimation laws can be written as:
\[
\begin{align*}
\dot{A} &= -P \, e \, x^T \, \Gamma_x \\
\dot{B} &= -P \, e \, u^T \, \Gamma_u 
\end{align*}
\]

12. System Identification using Dynamic Linear in Parameters Model and Output Feedback

It is clear that the parameter estimation laws (11.17) require full state measurement. Often in practice, only a subset of the system state components is sensed. These components, (called the on-board measurements), form the system measured output \( y \). Therefore, of interest are system ID methods that depend on the system output only.

Towards that end, consider the class of systems

\[
\begin{align*}
\dot{x} &= A \, x + B \, \Lambda \,(f_0(y,u) + \Theta^T \, \Phi(y,u)) \\
y &= C \, x 
\end{align*}
\] (12.1)

with known matrices \((A, B, C)\), known function \( f_0(y,u) \in R^m \), known regressor vector \( \Phi(y,u) \in R^m \), and unknown constant matrices \( \Lambda \in R^{m \times m} \) and \( \Theta \in R^{m \times N} \). The output vector \( y \in R^p \) represents all the on-board available measurements from the system. It is assumed that

\[
p = m < n
\] (12.2)

where \( m \) and \( n \) denote the dimensions of the control input \( u \) and the system state \( x \), respectively. In addition, it is assumed that \((A,B)\) is controllable, \((A,C)\) is observable, and the matrix triplet \((A, B, C)\) is Strictly Positive Real (SPR).

**Lemma 12.1 (Kalman-Yakubovich-Popov)**

Given a strictly proper, stable, rational transfer function \( W(s) \), assume that

\[
W(s) = C \left(s \, I_{nn} - A\right)^{-1} B
\] (12.3)

where \((A, B, C)\) is a minimal state-space realization of \( W(s) \) with \((A,B)\) controllable and \((A,C)\) observable. Then, \( W(s) \) is SPR if and only if there exists symmetric positive definite matrices \( P \) and \( Q \) such that
\begin{align}
\begin{cases}
PA + A^TP &= -Q \\
PB &= C^T
\end{cases} \tag{12.4}
\end{align}

**Remark 12.1**

The KYP Lemma and the SPR property immediately implies that $A$ is Hurwitz and the system is minimum phase.

**Remark 12.2**

In (12.1), the known triplet $(A, B, C)$ and the known possibly nonlinear function $f_0(y, u)$ represent the *nominal system dynamics*. The unknown constant matrix $\Lambda$ models *control failures*. Finally, the unknown quantity $\Theta^T \Phi(y, u)$ represents the system *matched uncertainty*.

The system dynamics (12.1) can be written as:

$$\dot{x} = A x + B \left( \Lambda \Theta^T \Phi(y, u) \right) = A x + B \bar{\Theta}^T \bar{\Phi}(y, u) \tag{12.5}$$

where $\Theta \in \mathbb{R}^{(N+m) \times m}$ denotes the extended matrix of unknown parameters, and $\Phi \in \mathbb{R}^{N+m}$ is the corresponding extended regressor vector. Thus the system takes the form:

$$\begin{cases}
\dot{x} = A x + B \bar{\Theta}^T \bar{\Phi}(y, u) \\
y = C x
\end{cases} \tag{12.6}$$

Based on (12.6), the *state predictor* is chosen as:

$$\begin{cases}
\dot{\hat{x}} = A \hat{x} + B \hat{\Theta}^T \Phi(y, u) \\
\hat{y} = C \hat{x}
\end{cases} \tag{12.7}$$

where $\hat{x} \in \mathbb{R}^n$ is the state of the predictor, $\hat{y} \in \mathbb{R}^p$ is its output, and $\hat{\Theta}^T = \left( \hat{\Lambda} \hat{\Lambda} \hat{\Theta}^T \right)$ is the extended matrix of estimated parameters.

Next, the *state prediction error* is introduced:

$$e = \hat{x} - x \tag{12.8}$$
Subtracting (12.6) from (12.7), the *prediction error dynamics* can be calculated:

\[
\begin{align*}
\dot{e} &= Ae + B \Delta \Phi^\top \Phi (y, u) \\
e_y &= C e 
\end{align*}
\]  

(12.9)

where the *output prediction error*

\[
e_y = \hat{y} - y
\]

(12.10)

is written as:

\[
e_y = \hat{y} - y = C(\hat{x} - x) = C e
\]

(12.11)

The *Lyapunov function candidate* is chosen in the form of (11.10), that is

\[
V(e, \Delta \Phi) = e^\top P e + \text{trace} \left( \Delta \Phi^\top \Phi + \Delta \Phi \Delta \Phi^\top \right)
\]

(12.12)

where \( P = P^\top > 0 \) and \( Q = Q^\top > 0 \) satisfy (12.4).

Differentiating (12.12) along the trajectories of the system (12.9), yields:

\[
\dot{V} = -e^\top Q e + 2 e^\top P B \Delta \Phi \Phi + 2 \text{trace} \left( \Delta \Phi^\top \Phi + \Delta \Phi \Delta \Phi^\top \right)
\]

(12.13)

\[
= -e^\top Q e + 2 e^\top \Phi \Delta \Phi \Phi + 2 \text{trace} \left( \Delta \Phi^\top \Phi + \Delta \Phi \Phi^\top \right)
\]

Using the trace identity (11.12), results in:

\[
\dot{V} = -e^\top Q e + 2 \text{trace} \left( \Delta \Phi^\top \left( \Phi \hat{\Phi} + \Phi \Phi^\top \right) \right)
\]

(12.14)

The *parameter estimation laws* are chosen as:

\[
\hat{\Phi} = -\Gamma \Phi (y, u) e_y^\top
\]

(12.15)

which leads to:

\[
\dot{V} = -e^\top Q e \leq 0
\]

(12.16)
Note that **full state information is not required** for on-line implementation of the estimation laws (12.15). In fact, the laws depend only on the **output** prediction error $e_y$ and the system input signal $u$.

From (12.16) it immediately follows that $V$ is the Lyapunov function for (12.9)-(12.15), and consequently:

$$
\begin{cases}
V, \hat{\Theta} \in L_\infty \\
\hat{\Theta}, e \in L_2 \cap L_\infty
\end{cases} \quad (12.17)
$$

Differentiating (12.16), one gets:

$$
\dot{V} = -e^T Q \dot{e} = -e^T Q \left( A e + B \Delta \bar{\Theta}^T \Phi \right) \quad (12.18)
$$

Hence, if $\Phi \in L_\infty$ then $\dot{V} \in L_\infty$ and, thus $V$ is a uniformly continuous function of time. The latter in combination with the fact that $(V \geq 0, \dot{V} \leq 0)$ implies (Barbalat’s Lemma) that $\dot{V} \to 0$, as $t \to \infty$. Because of (12.16), the system prediction error $e$ asymptotically tends to zero.

**Remark 12.3**

The estimation law (12.15) can be written in terms of the original system parameters from (12.5):

$$
\dot{\hat{\Theta}} = -\Gamma \left( f_0(y,u) \bigg\| \Phi(y,u) \right)(\hat{y} - y) \quad (12.19)
$$

The following theorem summarizes proven properties of the parameter estimation law (12.15).

**Theorem 12.1**

Given the dynamic linear in parameters model (12.1)

$$
\begin{align*}
\dot{x} &= A x + B \Lambda \left( f_0(y,u) + \Theta^T \Phi(y,u) \right) \\
y &= C x
\end{align*}
$$

$$
\begin{align*}
\dot{x} &= A x + B \bar{\Theta}^T \Phi(y,u) \\
y &= C x
\end{align*}
$$
suppose that \( x, u \in L_{\infty} \) and that the functions \( \{f_0(y,u), \Phi(y,u)\} \) are Lipschitz in \( y \).

Then using the state predictor dynamics (12.7)

\[
\begin{align*}
\dot{x} &= Ax + B\hat{\Theta}^T \Phi(y,u) \\
\hat{y} &= C\hat{x}
\end{align*}
\]

along with the parameter estimation law (12.19)

\[
\hat{\Theta} = -\Gamma \begin{bmatrix} f_0(y,u) \\ \Phi(y,u) \end{bmatrix} (\hat{y} - y)
\]

results in:

- \( e, \hat{\Theta} \in L_2 \)
- \( e, \hat{\Theta} \in L_{\infty} \)
- \( \lim_{t \to \infty} \|\hat{x}(t) - x(t)\| = 0 \)

If, in addition, the regressor vector \( \Phi \) is Persistently Exciting (PE) then the estimated parameters asymptotically converge to their true unknown constant values.

- \( \lim_{t \to \infty} \|\hat{\Theta}(t) - \Theta\| = 0 \)