Problem 1

(a) [Problem 2.2 in DFT]
For \( u_1 \):

For \( u_2 \):

For \( u_3 \): Note that \( \|u\|_1 = \|u\|_2^2 = \int_0^\infty 1 \cdot dt = \infty \), while \( \|u\|_{\infty} = 1 \) and

\[
\text{Pow}(u) = \lim_{T \to \infty} \frac{1}{2T} \int_0^T 1 \cdot dt = \frac{1}{2}
\]

For \( u_4 \):

For \( u_5 \):

For \( u_6 \): Note that in this case, \( \|u\|_1 = \|u\|_2 = \|u\|_{\infty} = \text{Pow}(u) = 0 \).

For \( u_7 \):

For \( u_8 \):

For \( u_9 \): We begin by noting that:

\[
\int_{-T}^T u^2(t)dt = \int_0^T u^2(t)dt = \int_0^T |u(t)|dt = \sum_{i=0}^{k-1} 2^{2i} \text{ for } 2^{2k-1} \leq T \leq 2^{2k}, k \geq 1.
\]

Hence, \( \|u\|_1 = \|u\|_2 = \infty \) while \( \|u\|_{\infty} = 1 \). Finally:

\[
\text{Pow}(u) = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^T u^2(t)dt = \begin{cases} 
\lim_{k \to \infty} \frac{1}{2^{2k}} \sum_{i=0}^{k-1} 2^{2i} = \frac{1}{4} & \text{ when } T = 2^{2k-1}, k \geq 1 \\
\lim_{k \to \infty} \frac{1}{2^{2k+1}} \sum_{i=0}^{k-1} 2^{2i} = \frac{1}{8} & \text{ when } T = 2^{2k}, k \geq 1
\end{cases}
\]

Hence the limit is not well-defined and \( u \) is not a power signal.
(b) [Problem 2.4 in DFT] Both norms are time-delay invariant:

\[
\| \hat{D} \hat{G} \|^2_2 = \frac{1}{2\pi} \int_{-\infty}^{\infty} (\hat{D} \hat{G})^*_jw (\hat{D} \hat{G})_jw \, dw
\]

\[
= \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{G}^*_jw \hat{D}^*_jw \hat{D} \hat{G} \, dw
\]

\[
= \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{G}^*_jw e^{jw\tau} e^{-jw\tau} \hat{G} \, dw
\]

\[
= \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{G}^*_jw \hat{G} \, dw
\]

\[
= \| \hat{G} \|^2_2
\]

and:

\[
\| \hat{D} \hat{G} \|_\infty = \sup_w |\hat{D} \hat{G}(jw)|
\]

\[
= \sup_w |e^{-jw\tau} \hat{G}(jw)|
\]

\[
= \sup_w |e^{-jw\tau}| \cdot |\hat{G}(jw)|
\]

\[
= \sup_w |\hat{G}(jw)|
\]

\[
= \| \hat{G} \|_\infty
\]

(c) [Problem 2.11 in DFT] Note that the given transfer function can be decomposed as:

\[
\hat{G}(s) = \frac{s + 2}{4s + 1} = \frac{1}{4} + \frac{7}{4} \frac{1}{4s + 1}
\]

The corresponding impulse response is:

\[
G(t) = \begin{cases} 
\frac{1}{4} \delta(t) + \frac{7}{16} e^{-\frac{1}{4}t} & t \geq 0 \\
0 & t < 0 
\end{cases}
\]

and

\[
\sup_{\|u\|_\infty = 1} \|y\|_\infty = \|G\|_1 = \int_{-\infty}^{\infty} |G(t)| \, dt = \frac{1}{4} + \frac{7}{16} \left(-4e^{-\frac{1}{4}t}\right) \bigg|_0^\infty = 2
\]

Finally, note that the unit step input achieves this bound.

(d) [Problem 2.13 in DFT] Consider \(\hat{G}_1(s) = \frac{1}{s + 1}\) and \(\hat{G}_2(s) = \frac{1}{s + 0.5}\), with corresponding 2-norms \(\|\hat{G}_1\|_2 = \frac{1}{\sqrt{2}}\) and \(\|\hat{G}_2\|_2 = 1\). We have

\[
\hat{G}_1(s) \hat{G}_2(s) = \frac{1}{(s + 1)(s + 0.5)} = \frac{2}{s + 0.5} - \frac{2}{s + 1}
\]
with corresponding impulse response \( G(t) = 2e^{-0.5t} - 2e^{-t}, \) for \( t \geq 0. \) Hence:

\[
\|\hat{G}_1 \hat{G}_2\|_2^2 = \|G\|_2^2 = 4 \int_0^\infty e^{-t} - 2e^{1.5t} + e^{-2t} \, dt = \frac{2}{3}
\]

Hence \( \|G_1 G_2\|_2 = \sqrt{\frac{2}{3}} > \frac{1}{\sqrt{2}} = \|G_1\|_2 \|G_2\|_2, \) which shows that the 2-norm is not submultiplicative.

Problem 2

(a) Recall that the equilibrium points are the solutions to the system of equations:

\[
\begin{align*}
\dot{x}_1 &= 0 \\
\dot{x}_2 &= 0
\end{align*}
\]

For \( S_1: \) Adding the two equations, we get \( x_2 = -x_1. \) Plugging back into either equation, we get a unique equilibrium point at the origin. Hence \( E_1 = \{(0,0)\}. \)

For \( S_2: \) Multiplying the first equation by \( x_2, \) multiplying the second equation by \( x_1, \) and adding, we get \( x_1^2 - x_2^2 = 0. \) Equivalently, \( x_2 = -x_1 \) or \( x_2 = x_1. \) When \( x_2 = x_1 = x, \) plugging back into either equation we get a unique equilibrium point at the origin. When \( x_2 = -x_1 = x, \) plugging back into either equation we get \( x(1 - 2x^2) = 0. \) It follows that the system has three equilibrium points:

\( E_2 = \{(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}), (0,0), (-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})\}. \)

For \( S_3: \) By inspection, it is clear that \( \dot{x}_1 = \dot{x}_2 = 0 \) whenever \( x_1 = 0 \) or \( x_2 = 0. \) Hence, the equilibrium points of the system are long the two coordinates axes: \( E_3 = \{(x_1, x_2)|x_1 = 0 \text{ or } x_2 = 0\}. \)

(b) Recall that \( V : \mathbb{R}^2 \rightarrow \mathbb{R} \) is a Lyapunov function for the system with equilibrium point \( x^* \) if:

(i) \( V(x) > V(x^*) \) for all \( x \in \mathbb{R}^2 \setminus \{x^*\} \)

(ii) \( \dot{V}(x) \leq 0 \) along all system trajectories in \( \mathbb{R}^2 \)

Thus, to show that none of the proposed functions is a Lyapunov function for \( S_1, \) it suffices to show that their derivative along system trajectories is strictly positive somewhere in \( \mathbb{R}^2 \setminus \{(0,0)\}. \)

Note that \( \dot{V}_a = 2x_1 \dot{x}_1 + 2x_2 \dot{x}_2. \) In particular, when \( x_2 = 0, \) the expression is a fourth order polynomial in \( x_1, \) with roots at \(-1, 1\) and double roots at 0. It is easy to check that \( \dot{V}_a > 0 \) whenever \( |x_1| > 1. \)

Also note that at \((2, -\frac{1}{2}), \) \( \dot{V}_b = -x_1 + x_2 + 2(x_1 + x_2)^3 = \frac{17}{4} > 0. \)

Finally, note that at \((2, 0), \) \( \dot{V}_c = -x_1 + (x_1 + x_2)^3 = 6 > 0. \)

Remark: Non-differentiability of a candidate Lyapunov function at certain points does not automatically disqualify it; non-positivity of subgradients at these
points may be sufficient, under certain conditions. However, this is beyond the scope of this class.

Problem 3

(a) Note that by definition, $V(x)$ is non-negative everywhere and 0 iff $x = 0$ since $P$ is positive definite. What is thus left is to require that $V$ be non-increasing along all system trajectories. Note that:

$$V(x(t+1)) - V(x(t)) = x'(t+1)Px(t+1) - x'(t)Px(t)$$

$$= x'(t)A'PAx(t) - x'(t)Px(t)$$

$$= x'(t)[A'PA - P]x(t)$$

We thus require $P$ to be a solution of the Lyapunov equation:

$$A'PA - P = -Q$$

for some symmetric positive semidefinite matrix $Q$.

(b) In this case, we require that $V$ be decreasing along all system trajectories, or equivalently, we require $P$ to be a solution of Lyapunov equation (1) for some symmetric positive definite matrix $Q$.

Problem 4

(a) Note that since $A$ is Hurwitz, for any choice of (symmetric positive matrix) $Q > 0$, there exists a $P > 0$ that is a solution to the Lyapunov equation:

$$A'P + PA = -Q$$

Moreover, $V(x) = x'Px$ is a Lyapunov function for the original system. The same candidate Lyapunov function can be used to show that the perturbed system is asymptotically stable provided that

$$\dot{V}(x) = x'(A + \Delta)'Px + x'P(A + \Delta)x < 0, \forall x$$

or equivalently:

$$(A + \Delta)'P + P(A + \Delta) < 0.$$

Now, note that every Hermitian matrix $H$ satisfies:

$$\sigma_{\text{min}}(H)I \leq H \leq \sigma_{\text{max}}(H)I$$

where the right hand side inequality was proved in problem 3 of Homework 1, and the left hand side inequality can be proved using a similar argument. It follows that:

$$\Delta'P + P\Delta - Q \leq \sigma_{\text{max}}(\Delta'P + P\Delta)I - \sigma_{\text{min}}(Q)I$$

$$= [\sigma_{\text{max}}(\Delta'P + P\Delta) - \sigma_{\text{min}}(Q)]I$$
Hence:
\[ \sigma_{\text{max}}(\Delta'P + P\Delta) - \sigma_{\text{min}}(Q) = -\epsilon < 0 \]  
(2)
is sufficient to guarantee that
\[ (A + \Delta)'P + P(A + \Delta) \leq -\epsilon I < 0 \]

What is left is to show that:
\[ \|\Delta\|_2 < \frac{\sigma_{\text{min}}(Q)}{2\sigma_{\text{max}}(P)} \]
implies (2). Note that:
\[ \sigma_{\text{max}}(\Delta'P + P\Delta) - \sigma_{\text{min}}(Q) = 2\sigma_{\text{max}}(P\Delta) - \sigma_{\text{min}}(Q) \]
\[ = 2\|P\Delta\|_2 - \sigma_{\text{min}}(Q) \]
\[ \leq 2\|P\|_2\|\Delta\|_2 - \sigma_{\text{min}}(Q) \]
\[ = 2\sigma_{\text{max}}(P)\|\Delta\|_2 - \sigma_{\text{min}}(Q) \]

which completes the argument.

(b) Let
\[ \Delta_o = \arg\min_{\Delta \in \mathbb{C}^{n \times n}} \left\{ \|\Delta\|_2 \mid A + \Delta \text{ is not Hurwitz} \right\} \]

We begin by noting that the eigenvalues of
\[ A + c\Delta_o \]
vary continuously in the complex plane as parameter \(c\) varies from 0 to 1. Thus, the eigenvalue of \(A + \Delta_o\) with maximal real part is exactly on the imaginary axis, say \(\lambda = jw_o\); for if that is not the case, a smaller destabilizing matrix \(c\Delta_o\), with \(c < 1\) can be found, which contradicts the assumption.

Also, note that it follows from the definition of eigenvalues that \(A + \Delta_o\) has an eigenvalue at \(jw_o\) iff the matrix \(A + \Delta_o - jw_oI\) is singular.

Next, note that for a fixed choice of \(w_o\), we have:
\[ \min_{\Delta \in \mathbb{C}^{n \times n}} \{ \|\Delta\|_2 \mid (A - jw_oI) + \Delta \text{ is singular} \} = \sigma_{\text{min}}(A - jw_oI) \]

To show that, suppose that \(A - jw_oI + \Delta\) is singular. Then there exists \(v \neq 0\), with \(\|v\|_2 = 1\), such that:
\[ ((A - jw_oI) + \Delta)v = 0 \iff \Delta v = -(A - jw_oI)v \]
\[ \Rightarrow \|\Delta v\|_2 = \|(A - jw_oI)v\|_2 \]

It thus follows that:
\[ \sigma_{\text{min}}(A - jw_oI) \leq \|(A - jw_oI)v\|_2 = \|\Delta v\|_2 \leq \|\Delta\|_2 \]

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where the first inequality was proved in problem 2 of Homework 1 and the second follows from the submultiplicative property of induced norms. What remains to show is that the lower bound can be achieved. With the singular value decomposition of $A - jw_o I = U \Sigma V^*$, let $\sigma = \sigma_{\min}(A - jw_o I)$ and consider:

$$\Delta = -\sigma u_n v_n^*$$

where $u_n$ and $v_n^*$ are the $n^{th}$ column of $U$ and $n^{th}$ row of $V^*$, respectively. Note that $\|\Delta\|_2 = \sigma$ and that:

$$(A - jw_o I + \Delta)v_n = U\Sigma^*v_n - \sigma u_n v_n^*v_n$$

$$= U\Sigma \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} - \sigma u_n$$

$$= U \begin{bmatrix} 0 \\ \vdots \\ 0 \\ \sigma \end{bmatrix} - \sigma u_n$$

$$= 0$$

Finally, what is left to note is that the choice of $w_o$ was arbitrary. Hence, we have:

$$\|\Delta_o\|_2 = \min_{w_o \in \mathbb{R}} \sigma_{\min}(A - jw I)$$

This provides a lower bound on $\gamma(A)$ since the set of complex square matrices of size $n$ contains the set of real square matrices of size $n$.

(c) When $\Delta$ is real, by an argument similar to that made in part (b), we have $A + \Delta$ is not Hurwitz iff $A - jw_o I + \Delta$ is singular for some $w_o \in \mathbb{R}$. Equivalently, there exists a $v \neq 0$ such that:

$$(A - jw_o I + \Delta)v = 0$$

By decomposing $v$ in terms of its real and imaginary parts as $v = \text{Re}(v) + j\text{Im}(v)$, plugging into the previous equation, and collecting real and imaginary parts we can rewrite this as:

$$(A + \Delta)\text{Re}(v) + w_o \text{Im}(v) = 0$$

$$(A + \Delta)\text{Im}(v) - w_o \text{Re}(v) = 0$$

which can again be equivalently re-written as:

$$\begin{bmatrix} A + \Delta & w_o I \\ -w_o I & A + \Delta \end{bmatrix} \begin{bmatrix} \text{Re}(v) \\ \text{Im}(v) \end{bmatrix} = 0 \iff \begin{bmatrix} A & w_o I \\ -w_o I & A \end{bmatrix} + \begin{bmatrix} \Delta & 0 \\ 0 & \Delta \end{bmatrix} \begin{bmatrix} \text{Re}(v) \\ \text{Im}(v) \end{bmatrix} = 0$$

$$\iff (A_{w_o} + \Delta) v = 0$$
By an argument similar to the one in part (b), it can be shown that for a fixed choice of \( w_0 \), and hence of \( A_w \) we have:

\[
\min_{\Delta \in \mathbb{R}^n \times n} \{ \| \Delta \|_2 | A_{w_0} + \Delta \text{ is singular} \} = \sigma_{\min}(A_{w_0})
\]

It follows from this and from the observation that the choice of \( w_0 \) was arbitrary that the smallest \textit{structured} (in this case \textit{block diagonal}) real perturbation \( \Delta \) that can destabilize the system satisfies:

\[
\| \Delta \|_2 \geq \min_{w \in \mathbb{R}} \sigma_{\min}(A_w)
\]

Finally, what is left to note is that \( \| \Delta \|_2 = \| \Delta \|_2 \), which can be readily verified using the singular value decomposition. Hence, \( \gamma(A) \geq \min_{w \in \mathbb{R}} \sigma_{\min}(A_w) \).

**Problem 5**

(a) We begin by noting that if a system is \( p \)-stable, then its response to an identically zero input, \( u(t) = 0, \forall t \), satisfies:

\[
\| \mathcal{H}(u) \|_p \leq C_o \| u \|_p = 0
\]

\[
\Rightarrow \mathcal{H}(u)(t) = 0, \forall t.
\]

For the given system, when \( u(t) = 0, \forall t \), we have \( x(t) = 0, \forall t \) and \( z(t) = 0, \forall t \) (assuming zero initial conditions, as stated in the problem). However, \( y(t) = \cos(0) = 1, \forall t \). Hence, the system cannot be \( p \)-stable for any choice of \( p \geq 1 \).

(b) We know that the linear map \( L \) with input \( u \) and output \( z \) is globally \( p \)-stable since \( A \) is Hurwitz. Hence, \( \exists C_L > 0 \) such that the inequality

\[
\| z \|_p \leq C_L \| u \|_p
\]

is satisfied for all inputs \( u \) and for any \( p \in \{1, 2, \infty\} \). For the nonlinear map \( g \), first note that since the input and output signals are scalar, we have \( \| z(t) \|_p = |z(t)| \) and \( \| y(t) \|_p = |y(t)| \) for any \( p \in \{1, 2, \infty\}, t \geq 0 \). Also note that it follows from the definition of \( g \) that \( |y(t)| \leq |z(t)| \). Hence, the inequality

\[
\| y \|_p \leq \| z \|_p
\]

follows for any input \( z \) and for any \( p \in \{1, 2, \infty\} \). Combining (3) and (4), we get:

\[
\| y \|_p \leq C_L \| u \|_p
\]

and thus the system is \textit{globally} \( p \)-stable for any \( p \in \{1, 2, \infty\} \).

(c) By an argument similar to that made in part (b), and noting that \( |\sin(x)| \leq |x| \) for all \( x \), the system is globally \( p \)-stable for any \( p \in \{1, 2, \infty\} \).