Problem 1

(a) Recall that the null space and the range of a matrix $A \in \mathbb{C}^{m \times n}$ are defined as:

$$
\mathcal{N}(A) = \{x \in \mathbb{C}^n | Ax = 0\}
$$

$$
\mathcal{R}(A) = \{y \in \mathbb{C}^m | y = Ax, \text{ for some } x \in \mathbb{C}^n\}
$$

Note that $\mathcal{N}(A)$ is a subspace of $\mathbb{C}^n$ since it contains $0 \in \mathbb{C}^n$ ($A0 = 0$) and is closed under addition and scalar multiplication:

$$
\begin{align*}
\text{if } x_1, x_2 \in \mathcal{N}(A) & \Rightarrow Ax_1 = 0 \text{ and } Ax_2 = 0 \\
& \Rightarrow Ax_1 + Ax_2 = 0 \\
& \Rightarrow A(x_1 + x_2) = 0 \\
& \Rightarrow x_1 + x_2 \in \mathcal{N}(A)
\end{align*}
$$

$$
\begin{align*}
x \in \mathcal{N}(A), \alpha \in \mathbb{C} & \Rightarrow A(\alpha x) = \alpha Ax = 0 \Rightarrow \alpha x \in \mathcal{N}(A)
\end{align*}
$$

Similarly, note that $\mathcal{R}(A)$ is a subspace of $\mathbb{C}^m$ since it contains $0 \in \mathbb{C}^m$ ($A0 = 0$) and is closed under addition and scalar multiplication:

$$
\begin{align*}
y_1, y_2 \in \mathcal{R}(A) & \Rightarrow y_1 = Ax_1 \text{ and } y_2 = Ax_2 \text{ for some } x_1, x_2 \in \mathbb{C}^n \\
& \Rightarrow y_1 + y_2 = Ax_1 + Ax_2 = A(x_1 + x_2) = Ax_3 \text{ for some } x_3 \in \mathbb{C}^n \\
& \Rightarrow y_1 + y_2 \in \mathcal{R}(A)
\end{align*}
$$

$$
\begin{align*}
y \in \mathcal{R}(A), \alpha \in \mathbb{C} & \Rightarrow y = Ax \text{ for some } x \in \mathbb{C}^n, \alpha y = \alpha Ax = A(\alpha x) \\
& \Rightarrow \alpha y \in \mathcal{R}(A)
\end{align*}
$$
(b) Note that:

\[ x \in \mathcal{R}^\perp(A) \iff x^*Ay = 0, \forall y \in \mathbb{C}^n \]
\[ \iff y^*Ax = 0, \forall y \in \mathbb{C}^n \]
\[ \iff A^*x = 0 \]
\[ \iff x \in \mathcal{N}(A^*) \]

(c) \( \mathcal{N}^\perp(A) = \mathcal{R}(A^*) \) follows from part (b) and the following linear algebra facts. Let \( V \) and \( W \) be subspaces of an inner product space \((X, <, >)\); we have:

- \((V^\perp)^\perp = V\), for all \( V \).
- \( V = W \iff V^\perp = W^\perp \).

(d) Let \( r = \dim(\mathcal{N}(A)) \), \( B_1 = \{v_1, v_2, \ldots, v_r\} \) be a set of basis vectors for \( \mathcal{N}(A) \), and \( B_2 = \{v_{r+1}, \ldots, v_n\} \) be a set of vectors such that \( B = B_1 \cup B_2 \) is a basis for \( \mathbb{C}^n \). We will prove that \( \dim(\mathcal{R}(A)) = n - r \).

We begin by noting that \( Av_{r+1}, Av_{r+2}, \ldots, Av_n \) are linearly independent, for if that was not the case, then there exists scalars \( a_{r+1}, a_{r+2}, \ldots, a_n \) (not all zero) such that:

\[ a_{r+1}Av_{r+1} + a_{r+2}Av_{r+2} + \ldots + a_nAv_n = 0 \implies A(a_{r+1}v_{r+1} + a_{r+2}v_{r+2} + \ldots + a_nv_n) = 0 \]
\[ \implies a_{r+1}v_{r+1} + a_{r+2}v_{r+2} + \ldots + a_nv_n \in \mathcal{N}(A) \]

leading to a contradiction. Thus, \( \dim(\mathcal{R}(A)) \geq n - r \) since there are at least \( n - r \) linearly independent vectors in \( \mathcal{R}(A) \).

Now suppose there exists \( y \in \mathcal{R}(A) \) such that \( y, Av_{r+1}, Av_{r+2}, \ldots, Av_n \) are linearly independent. Then, there exists \( x \in \mathbb{C}^n \) such that \( y = Ax \) and \( x \notin \text{span}(B_2) \), again leading to a contradiction. Thus \( \dim(\mathcal{R}(A)) = n - r \).

**Problem 2**

(a) Recall that the 1-induced norm of a \( A \) is defined as:

\[ \|A\|_1 = \sup_{x \neq 0} \frac{\|Ax\|_1}{\|x\|_1} = \max_{\|x\|_1 = 1} \|Ax\|_1 \]
We begin by showing that $\|A\|_1 \leq \max_{1 \leq j \leq n} \sum_{i=1}^m |a_{ij}|$. We have:

$$
\|Ax\|_1 = \sum_{i=1}^m \left| \sum_{j=1}^n a_{ij} x_j \right|
\leq \sum_{i=1}^m \sum_{j=1}^n |a_{ij} x_j|
= \sum_{i=1}^m \sum_{j=1}^n |a_{ij}||x_j|
= \sum_{j=1}^n \left( |x_j| \sum_{i=1}^m |a_{ij}| \right)
\leq \sum_{j=1}^n \left( |x_j| \cdot \max_{1 \leq j \leq n} \sum_{i=1}^m |a_{ij}| \right)
= \max_{1 \leq j \leq n} \sum_{i=1}^m |a_{ij}| \cdot \sum_{1 \leq j \leq n} |x_j|
= \max_{1 \leq j \leq n} \sum_{i=1}^m |a_{ij}| \cdot \|x\|_1
$$

To show that this upper bound can be achieved, let $j_o = \arg\max_j \sum_{i=1}^m |a_{ij}|$, and consider $\hat{x} = e_{j_o}$, the $j^\text{th}$ basis vector (i.e. $\hat{x}_i = 1$ when $i = j_o$ and 0 otherwise). Clearly, $\|\hat{x}\|_1 = 1$ and:

$$
Ax = \begin{bmatrix}
a_{1j_o} \\
\vdots \\
a_{mj_o}
\end{bmatrix} \Rightarrow \|Ax\|_1 = \sum_{i=1}^m |a_{ij_o}|
$$

which completes the proof.

(b) Recall that the $\infty$-induced norm of $A$ is defined as:

$$
\|A\|_\infty = \sup_{x \neq 0} \frac{\|Ax\|_\infty}{\|x\|_\infty} = \max_{\|x\|_\infty = 1} \|Ax\|_\infty
$$
We begin by showing that $\|A\|_\infty \leq \max_{1 \leq i \leq m} \sum_{j=1}^{n} |a_{ij}|$. We have:

$$
\|Ax\|_\infty = \max_{1 \leq i \leq m} \left| \sum_{j=1}^{n} a_{ij}x_j \right|
\leq \max_{1 \leq i \leq m} \sum_{j=1}^{n} |a_{ij}x_j|
= \max_{1 \leq i \leq m} \sum_{j=1}^{n} |a_{ij}| \|x_j\|
\leq \max_{1 \leq i \leq m} \left( \sum_{j=1}^{n} |a_{ij}| \cdot \left( \max_{1 \leq j \leq n} |x_j| \right) \right)
= \max_{1 \leq i \leq m} \left( \sum_{j=1}^{n} |a_{ij}| \cdot \|x\|_\infty \right)
= \left( \max_{1 \leq i \leq m} \sum_{j=1}^{n} |a_{ij}| \right) \cdot \|x\|_\infty
= \max_{1 \leq i \leq m} \sum_{j=1}^{n} |a_{ij}| \text{ whenever } \|x\|_\infty = 1
$$

To show that this upper bound can be achieved, let $i_o = \arg\max_i \sum_{j=1}^{n} |a_{ij}|$ and consider vector $\hat{x}$ defined as:

$$
\hat{x} = \begin{bmatrix}
\sgn(a_{i_o,1}) \\
\vdots \\
\sgn(a_{i_o,n})
\end{bmatrix}
$$

Clearly $\|\hat{x}\|_\infty = 1$ since $|\hat{x}_j| = 1$ for all $j$, and

$$
\|A\hat{x}\|_\infty = \sum_{j=1}^{n} a_{i_o,j} \sgn(a_{i_o,j}) = \sum_{j=1}^{n} |a_{i_o,j}| = \max_{1 \leq i \leq m} \sum_{j=1}^{n} |a_{ij}|
$$

which completes the proof.

(c) We begin by noting that the inequality

$$
\|A\|_\infty \leq \|A\|_2 \leq \sqrt{m}\|A\|_\infty
$$

holds when $n = 1$ (i.e. when $A$ is a vector) since:

$$
\|A\|_\infty^2 = \left( \max_{1 \leq i \leq m} |a_i| \right)^2 \leq \sum_{i=1}^{m} |a_i|^2 = \|A\|_2^2
$$
and
\[ \|A\|^2 = \sum_{i=1}^{m} |a_i|^2 \leq \sum_{i=1}^{m} \left( \max_{1 \leq i \leq m} |a_i| \right)^2 = m\|A\|^2_{\infty} \]

When \( x \neq 0 \), we have:
\[ \frac{\|Ax\|_\infty}{\|x\|_\infty} \geq \frac{\|Ax\|_\infty}{\|x\|_2} \geq \frac{1}{\sqrt{m}} \frac{\|Ax\|_2}{\|x\|_2} \]

with each of the inequalities following directly from the established vector inequalities. Hence:
\[ \|A\|_\infty = \sup_{x \neq 0} \frac{\|Ax\|_\infty}{\|x\|_\infty} \geq \frac{1}{\sqrt{m}} \frac{\|Ax\|_2}{\|x\|_2} \]

for all \( x \neq 0 \), from which it follows that:
\[ \|A\|_\infty = \sup_{x \neq 0} \frac{\|Ax\|_\infty}{\|x\|_\infty} \geq \frac{1}{\sqrt{m}} \frac{\|Ax\|_2}{\|x\|_2} = \frac{1}{\sqrt{m}} \|Ax\|_2 \]

which proves the right hand side inequality.

When \( x \neq 0 \), it also follows from the established vector identities that \( \|Ax\|_\infty \leq \|Ax\|_2 \). Hence we have:
\[ \sqrt{n} \frac{\|Ax\|_\infty}{\|x\|_2} \leq \sqrt{n} \frac{\|Ax\|_2}{\|x\|_2} \leq \sup_{x \neq 0} \sqrt{n} \frac{\|Ax\|_2}{\|x\|_\infty} = \sqrt{n} \|A\|_2 \]

Noting that for \( x \in \mathbb{C}^n \) the established vector inequality implies that \( \frac{1}{\|x\|_\infty} \leq \frac{\sqrt{n}}{\|x\|_2} \), we have:
\[ \frac{\|Ax\|_\infty}{\|x\|_\infty} \leq \frac{\sqrt{n}}{\|x\|_2} \leq \sqrt{n} \|A\|_2 \Rightarrow \|A\|_\infty = \sup_{x \neq 0} \frac{\sqrt{n}}{\|x\|_\infty} \|A\|_2 \leq \sqrt{n} \|A\|_2 \]

which proves the left hand side inequality.

(d) With the inherent assumption that \( m \geq n \), suppose that \( \text{rank}(A) = n \) and let \( U \Sigma V^* \) be the singular value decomposition of \( A \). We have:
\[ \|Ax\|_2^2 = \|U \Sigma V^* x\|_2^2 \]
\[ = \|\Sigma V^* x\|_2^2 \quad \text{(Since } U \text{ is unitary)} \]
\[ = \|\Sigma y\|_2^2 \text{ where } y = V^* x \]
\[ = \sum_{i=1}^{n} |\sigma_i y_i|^2 \]
\[ \geq \sigma_{\min}(A)^2 \|y\|_2^2 \]
\[ = \sigma_{\min}(A)^2 \|V^* x\|_2^2 \]
\[ = \sigma_{\min}(A)^2 \|x\|_2^2 \quad \text{(Since } V \text{ is unitary)} \]
Where $e \in \mathbb{C}$ of Problem 3, $x = v_n$, where $v_n$ is the $n^{th}$ column of $V$. Then:

$$V^*x = \begin{bmatrix}
v_1^* \\
\vdots \\
v_{n-1}^* \\
v_n^*
\end{bmatrix}v_n = \begin{bmatrix}0 \\
\vdots \\
0 \\
1
\end{bmatrix} = e_n,$$

$$\Sigma V^*x = \Sigma e_n = \sigma_{\min}(A)e_m$$

where $e_m$ is the basis vector with unity entry in row $m$, and

$$Ax = U\Sigma V^*x = U\sigma_{\min}(A)e_m = \sigma_{\min}(A)u_m$$

where $u_m$ is the $m^{th}$ column of $U$. Hence:

$$\|Ax\|_2 = \|\sigma_{\min}(A)u_m\|_2 = \sigma_{\min}(A)\|u_m\|_2 = \sigma_{\min}(A).$$

When $\text{rank}(A) < n$, note that $Ax = 0$ for $x = v_n$ since $\Sigma V^*x = 0$. Hence $\min_{\|x\|_2=1}\|Ax\|_2 = 0$ in this case.

**Problem 3**

(a) Let $x = \alpha_{k+1}v_{k+1} + \ldots + \alpha_nv_n$ for some $\alpha_{k+1}, \ldots, \alpha_n \in \mathbb{C}$. We have:

$$\|Ax\|_2 = \|U\Sigma V^*x\|_2 = \|\Sigma V^*x\|_2$$

since $U$ is unitary, and:

$$\Sigma V^*x = \begin{bmatrix}
v_1^* \\
\vdots \\
v_{n-1}^* \\
v_n^*
\end{bmatrix} = \Sigma \begin{bmatrix}0 \\
\vdots \\
0 \\
\alpha_{k+1} \\
\vdots \\
\alpha_n
\end{bmatrix} = \begin{bmatrix}0 \\
\vdots \\
0 \\
\sigma_{k+1}\alpha_{k+1} \\
\vdots \\
\sigma_n\alpha_n
\end{bmatrix}$$

since $V$ is unitary, hence $v_i^*v_j = 1$ whenever $i = j$ and is 0 otherwise. Thus:

$$\|Ax\|_2^2 = \sum_{i=k+1}^n |\alpha_i\sigma_i|^2 \leq \sum_{i=k+1}^n |\alpha_i|^2\sigma_{k+1}^2$$

and

$$\|x\|_2^2 = \|\alpha_{k+1}v_{k+1}^* + \ldots + \alpha_nv_n^*\|_2^2 = \|\alpha_{k+1}v_{k+1}^* + \ldots + \alpha_n v_n^*(\alpha_{k+1}v_{k+1} + \ldots + \alpha_nv_n)\|_2^2 = \sum_{i=k+1}^n |\alpha_i|^2$$
Hence $\|Ax\|_2 \leq \sigma_{k+1}\|x\|_2$.

(b) We have:

\[ A \text{ is invertible } \iff \det(A) \neq 0 \]
\[ \iff \det(U\Sigma V^*) \neq 0 \]
\[ \iff \det(U)\det(\Sigma)\det(V^*) \neq 0 \]
\[ \iff \det(\Sigma) \neq 0 \]
\[ \iff \sigma(A) \neq 0 \]
\[ \iff \sigma(A) > 0 \]

where the third equivalence follows because $U$, $\Sigma$ and $V^*$ are square matrices of identical sizes, and where the fourth equivalence follows because both $U$ and $V$ are unitary hence invertible.

Note that when $A$ is invertible, $A^{-1}$ is given by

\[ A^{-1} = V\Sigma^{-1}U^* \]

where

\[
\Sigma^{-1} = \begin{bmatrix}
\frac{1}{\sigma(A)} & 0 & \ldots & 0 \\
0 & \ddots & \vdots \\
\vdots & \ddots & \ddots & 0 \\
0 & \ldots & 0 & \frac{1}{\sigma(A)}
\end{bmatrix}
\]

Hence, the singular values of $A^{-1}$ are the reciprocals of the singular values of $A$ with

\[ \sigma(A^{-1}) = \frac{1}{\sigma(A)} \]

(c) Recall that a matrix is symmetric positive semidefinite iff all its eigenvalues are non-negative, and that the singular values of a Hermitian matrix are identical to its eigenvalues. Also, note that the singular value decomposition of a Hermitian matrix is of the form $U\Sigma U^*$. We have:

\[ \sigma(A)I - A = \sigma(A)I\Sigma U^* - U\Sigma U^* = U(\sigma(A)I - \Sigma)U^* \]

The eigenvalues of $\sigma(A)I - A$ are then given by $0, \sigma(A) - \sigma_2(A), \ldots, \sigma(A) - \sigma(A)$ and are thus all non-negative, which proves that:

\[ \sigma(A)I - A \geq 0 \iff A \leq \sigma(A)I \]

Similarly we have:

\[ A + \sigma(A)I = U(\Sigma + \sigma I)U^* \]

and the eigenvalues of $A + \sigma(A)I$ are clearly non-negative, each being the sum of two non-negative terms. Hence:

\[ A + \sigma(A)I \geq 0 \iff -\sigma(A) \leq A. \]
Problem 4

• $\sup_t |\dot{u}(t)|$ is not a norm as it doesn’t satisfy the positivity condition:
  Consider a constant signal $u(t) = c > 0$, and note that $\dot{u}(t) = 0$ for all $t$, hence $\sup_t |\dot{u}(t)| = 0$ even though $u(t)$ is a non-zero signal.

• $\|u\| = |u(0)| + \sup_t |\dot{u}(t)|$ qualifies as a norm:
  (i) (Positivity) Being the sum of nonnegative quantities, $\|u\| \geq 0$ for all $u$. Moreover
  \[
  \|u\| = 0 \iff \begin{cases} |u(0)| = 0 \\ |\dot{u}(t)| = 0 \quad \forall t \geq 0 \end{cases} \iff u(t) = 0, \text{ for all } t \geq 0
  \]
  (ii) (Homogeneity) Note that for any scalar $\alpha$, we have:
  \[
  \|\alpha u\| = |\alpha u(0)| + \sup_t \left| \frac{d(\alpha \dot{u}(t))}{dt} \right| = |\alpha| |u(0)| + \sup_t |\alpha \dot{u}(t)| = |\alpha| \cdot \|u\|
  \]
  (iii) (Triangle Inequality) Note that:
  \[
  \|u + v\| = |u(0) + v(0)| + \sup_t \left| \frac{d(u(t) + v(t))}{dt} \right| \\
  \leq |u(0)| + |v(0)| + \sup_t |\dot{u}(t) + \dot{v}(t)| \\
  \leq |u(0)| + |v(0)| + \sup_t |\dot{u}(t)| + \sup_t |\dot{v}(t)| \\
  = \|u\| + \|v\|
  \]

• $\|u\| = \max\{\sup_t |u(t)|, \sup_t |\dot{u}(t)|\}$ qualifies as a norm:
  (i) (Positivity) Being the sum of nonnegative quantities, $\|u\| \geq 0$ for all $u$. Moreover
  \[
  \|u\| = 0 \iff \begin{cases} \sup_t |u(t)| = 0 \\ \sup_t |\dot{u}(t)| = 0 \end{cases} \iff u(t) = 0, \text{ for all } t \geq 0
  \]
  (ii) (Homogeneity) Follows from the homogeneity of the supremum (see above) and from the fact that $\max\{\alpha f_1, \alpha f_2\} = \alpha \max\{f_1, f_2\}$.
  (iii) (Triangle Inequality) Having already established that:
  \[
  \sup_t |u(t) + v(t)| \leq \sup_t |u(t)| + \sup_t |v(t)| \\
  \sup_t |\dot{u}(t) + \dot{v}(t)| \leq \sup_t |\dot{u}(t)| + \sup_t |\dot{v}(t)|
  \]
what is left to note is that:

$$\max\{f_1 + f_2, f_3 + f_4\} \leq \max\{f_1, f_3\} + \max\{f_2, f_4\}$$

The triangle inequality then follows.

- $\|u\| = \sup_t |u(t)| + \sup_t |\dot{u}(t)|$ qualifies as a norm:
  
  
  (i) (Positivity) Being the sum of nonnegative quantities, $\|u\| \geq 0$ for all $u$. Moreover
  
  $$\|u\| = 0 \iff \left\{ \begin{array}{l} \sup_t |u(t)| = 0 \\ \sup_t |\dot{u}(t)| = 0 \end{array} \right. \iff u(t) = 0, \text{ for all } t \geq 0$$
  
  (ii) (Homogeneity) Follows from the homogeneity of the supremum.
  
  (iii) (Triangle Inequality) Follows from the fact that both $\sup_t |u(t)|$ and $\sup_t |\dot{u}(t)|$ satisfy the triangle inequality, and hence so does their sum.