Problem 1

(a) Recall that the $H_\infty$ norm of a transfer function is time-delay invariant. Hence:

$$\|\hat{G}(s)\|_\infty = \|\frac{1}{s + a}\|_\infty = \sup_{w \in \mathbb{R}} \left|\frac{1}{jw + a}\right| = \sup_{w \in \mathbb{R}} \left(\frac{1}{a^2 + w^2}\right)^{1/2} = \frac{1}{a}$$

(b) The transfer function of the system is $\hat{G}(s) = \frac{1}{s + 1}$, with corresponding $H_2$ norm:

$$\|\hat{G}\|_2^2 = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{G}^*(jw)\hat{G}(jw)dw$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{1}{1 - jw} \frac{1}{1 + jw}dw$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{1}{1 + w^2}dw$$

$$= \frac{1}{2\pi} \arctan w \bigg|_{-\infty}^{\infty}$$

$$= \frac{1}{2}$$

hence $\|\hat{G}\|_2 = \frac{1}{\sqrt{2}}$

(c) Recall that:

$$\|y\|_\infty \leq \|G\|_1 \|u\|_\infty$$

where $G(t)$ is the impulse response of the system. Here, $G(t) = te^{-t}$ when
\[ \dot{x}_1 = \frac{x}{s-1} \quad \dot{x}_2 = \frac{1}{s+2} \]

Figure 1: Figure for problem 1(d)

\[ t \geq 0, \text{ with corresponding } l_1 \text{ norm:} \]

\[ \|G\|_1 = \int_{-\infty}^{\infty} |G(t)| dt \]

\[ = \int_{0}^{\infty} te^{-t} dt \]

\[ = -e^{-t}\bigg|_{0}^{\infty} + \int_{0}^{\infty} e^{-t} dt \]

\[ = 0 - e^{-t}\bigg|_{0}^{\infty} \]

\[ = 1 \]

and the amplitude of the output cannot exceed 1.

(d) Consider the parallel interconnection of two first order systems as shown in the Figure above, and note by inspection that \( \hat{G} \) is indeed the corresponding transfer matrix.

Let \( x_0(t) = L^{-1}(\hat{x}_0(s)) \) and \( x_2(t) = L^{-1}(\hat{x}_2(s)) \). We have:

\[ \dot{x}_2 + 2x_2 = u_2 \]

\[ \dot{x}_0 - x_0 = \dot{u}_1 \]

\[ y = x_0 + x_2 \]

Set \( x_1 = x_0 - u_1 \), with \( \dot{x}_1 = \dot{x}_0 - \dot{u}_1 = x_0 = x_1 + u_1 \). The state-space equations of the system are then:

\[
\begin{cases}
    \dot{x}_1 &= \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \\
y &= \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}
\end{cases}
\]

(e) For the autonomous system (i.e. input identically 0), the state and output trajectories for \( t \geq 0 \) are given by:

\[ x(t) = e^{At}x(0) \]

\[ y(t) = Ce^{At}x(0) \]
where $e^{At}$ is defined by the infinite series:

$$e^{At} = I + At + \frac{A^2 t^2}{2!} + \frac{A^3 t^3}{3!} + \ldots$$

Note that here:

$$A^2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad A^k = 0, \quad \text{for} \quad k \geq 3$$

Hence

$$e^{At} = \begin{bmatrix} 1 & t & t^2/2 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{bmatrix}$$

and

$$x(t) = \begin{bmatrix} 1 + 2t + t^2/2 \\ 2 + t \\ 1 \end{bmatrix},$$

$$y(t) = 1 + 2t + t^2/2,$$

for $t \geq 0$.

(f) The transfer matrix is given by $\hat{G}(s) = C(sI - A)^{-1}B + D$. Note that due to the structure of $C$ and $B$, we only need to compute the upper right entry of $(sI - A)^{-1}$. Recall that

$$(sI - A)^{-1} = \frac{\text{adj}(sI - A)}{\det(sI - A)}$$

We have:

$$sI - A = \begin{bmatrix} s & -1 & 0 \\ 0 & s & -1 \\ 0 & 0 & s \end{bmatrix},$$

$$\text{adj}(sI - A) = \begin{bmatrix} * & * & * \\ * & * & * \\ 1 & * & * \end{bmatrix}'$$

and

$$\det(sI - A) = s \left| \begin{array}{ccc} s & -1 & 0 \\ 0 & s & -1 \\ 0 & 0 & s \end{array} \right| + 1 \left| \begin{array}{ccc} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & s \end{array} \right| = s^3$$

Hence

$$\hat{G}(s) = \begin{bmatrix} 1 & 0 & 0 \\ * & * & \frac{1}{s^3} \\ * & * & * \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + 1 = \frac{s^3 + 1}{s^3}$$
Problem 2

(a) FALSE. Consider a SISO system with $A=B=1, C=D=0$. $A$ has an eigenvalue at 1 while the transfer function is identically 0 and hence has no poles - this is a very trivial example of an unobservable system.

(b) TRUE. We have:

$$\|\hat{G}_1\|_2^2 = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{G}^*(rjw)\hat{G}(rjw)dw$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{G}^*(jv)\hat{G}(jv)\frac{1}{r}dv$$

$$= r^{-1}\|\hat{G}\|_2^2$$

where the second equality follows by a change of the integration variable.

(c) FALSE. Consider the case where $M = aI$ for some scalar $|a| < 1$. Clearly, there exists a matrix $D = I = D^{-1}$ such that:

$$\|DMD^{-1}\|_2 = \|M\|_2 = |a| \cdot \|I\|_2 = |a| < 1$$

However, the LMI:

$$\begin{bmatrix} -X & 0 \\ 0 & a^*IXIa + X \end{bmatrix} = \begin{bmatrix} -X & 0 \\ 0 & (|a|^2 + 1)X \end{bmatrix} < 0$$

is not feasible, since it requires $X$ to be both positive definite and negative definite. The correct statement will be derived later on in the class!

(d) TRUE. One way of proving this is by using state-space methods to compute the $\mathcal{H}_2$ norms of the relevant systems (see Section 2.6 in DFT). Possible state-space realizations of systems $S_1$ and $S_2$ described by transfer functions $\hat{G}_1$ and $\hat{G}_2$ are given by:

$$S_1 = \begin{pmatrix} -a_1 & 1 \\ 1 & 0 \end{pmatrix}, S_2 = \begin{pmatrix} -a_2 & 1 \\ 1 & 0 \end{pmatrix}$$

System $S$, the cascade interconnection of $S_1$ and $S_2$, with corresponding transfer function $\hat{G}(s) = \hat{G}_2(s)\hat{G}_1(s)$, then has the following state space realization:

$$S = \begin{bmatrix} -a_1 & 0 \\ 1 & -a_2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Recall that for a system

$$S = \begin{pmatrix} A \\ C \end{pmatrix} \begin{pmatrix} B \\ D \end{pmatrix}$$
with $A$ Hurwitz, the $\mathcal{H}_2$ norm of the corresponding transfer function $\hat{G}$ can be computed as:

$$||\hat{G}||_2^2 = CLC'$$

where $L$ is the solution to the Lyapunov equation:

$$AL + LA' = -BB'$$

For first order systems $S_1$ and $S_2$, we thus have:

$$||\hat{G}_i||_2^2 = -\frac{c_i^2 b_i^2}{2a_i}$$

For the second order system $S$, the relevant Lyapunov equation is:

$$\begin{bmatrix} -a_1 & 0 \\ 1 & -a_2 \end{bmatrix} \begin{bmatrix} l_1 & l_0 \\ l_0 & l_2 \end{bmatrix} + \begin{bmatrix} l_1 & l_0 \\ l_0 & l_2 \end{bmatrix} \begin{bmatrix} -a_1 & 1 \\ 0 & -a_2 \end{bmatrix} = -\begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \end{bmatrix}$$

Solving the corresponding system of three equations in three unknowns:

$$\begin{aligned}
-2a_1 l_1 + 1 &= 0 \\
-(a_1 + a_2) l_0 + l_1 &= 0 \\
l_0 - a_2 l_2 &= 0
\end{aligned}$$

we get:

$$l_1 = \frac{1}{2a_1}, \quad l_0 = \frac{1}{2a_1(a_1 + a_2)}, \quad l_2 = \frac{1}{2a_1a_2(a_1 + a_2)}$$

Hence $||\hat{G}||_2^2 = l_2$, and

$$\frac{1}{2a_1a_2(a_1 + a_2)} \leq \frac{-1}{2a_1} \cdot \frac{-1}{2a_2} \Rightarrow a_1 a_2 \geq 2.$$
where
\[
\tilde{\Sigma} = \begin{bmatrix}
\frac{1}{\sigma_{\max}(A)} & 0 & \ldots & 0 \\
0 & 0 & \ddots & \\
\vdots & \ddots & \ddots & \\
0 & \ldots & \ldots & 0
\end{bmatrix}
\]

Note that \( \|\Delta\|_2 = \frac{1}{\sigma_{\max}(A)} \) and that:

\[
I - \Delta A = I - V\tilde{\Sigma}U^*U\Sigma V^* = I - V\tilde{\Sigma}\Sigma V^* = V\begin{pmatrix}
1 & 0 & \ldots & 0 \\
0 & \ddots & \ddots & \\
\vdots & \ddots & \ddots & \\
0 & \ldots & \ldots & 0
\end{pmatrix}V^*
\]

with \((I - \Delta A)v_1 = 0\); hence \(I - \Delta A\) is singular.
Problem 3

(a) Let $x_1 = y$ and $x_2 = \dot{y} - u$. We then have:

$$\begin{align*}
\dot{x}_1 & = \dot{y} \\
& = x_2 + u
\end{align*}$$

and

$$\begin{align*}
\dot{x}_2 & = \ddot{y} - u \\
& = -a\dot{y} - by - cy_2 + u \\
& = -a(x_2 + u) - bx_1 - cx_1^2 + u \\
& = -bx_1 - ax_2 - cx_1^2 + (1 - a)u
\end{align*}$$

The corresponding second order state space realization is:

$$\begin{align*}
\dot{x}_1 & = x_2 + u \\
\dot{x}_2 & = -bx_1 - ax_2 - cx_1^2 + (1 - a)u \\
y & = x_1
\end{align*}$$

Note that this realization is not unique; it is simply one of many possible realizations.

(b) When $a = 3$, $b = 2$ and $c = 2$, the state equations of the autonomous system reduce to:

$$\begin{align*}
\dot{x}_1 & = x_2 \\
\dot{x}_2 & = -2x_1 - 3x_2 - 2x_1^2
\end{align*}$$

and the corresponding linearized (about the origin) dynamics are given by:

$$\begin{pmatrix}
\dot{\delta}_1 \\
\dot{\delta}_2
\end{pmatrix} =
\begin{bmatrix}
0 & 1 \\
-2 & -3
\end{bmatrix}
\begin{pmatrix}
\delta_1 \\
\delta_2
\end{pmatrix}$$

The eigenvalues of the linearized dynamics are $\lambda_1 = -1$ and $\lambda_2 = -2$. Since the linearized dynamics are asymptotically stable, and for $f(x) = -2x^2$, we have:

$$\begin{align*}
\lim_{\|\delta\|_2 \to 0} \frac{\|f(\delta) - f(0) - f'(0)\delta\|_2}{\|\delta\|_2} &= \lim_{\|\delta\|_2 \to 0} \frac{2|\delta^2|}{\|\delta\|_2} = 0
\end{align*}$$

we conclude that the equilibrium point at the origin is locally asymptotically stable. Clearly, it cannot be globally stable as there exists another equilibrium point at $(−1, 0)$. 

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(c) The dynamics of the autonomous nonlinear system are given by:

\[ \dot{x} = Ax + f(x) \]

where

\[ A = \begin{bmatrix} 0 & -2 \\ 1 & -3 \end{bmatrix} \quad \text{and} \quad f(x) = \begin{bmatrix} 0 \\ -2x_1^2 \end{bmatrix} \]

Let \( V(x) = x'Px \) be a Lyapunov function for the linearized system; then \( P \) is a solution of the Lyapunov equation \( A'P + PA = -Q \) for some \( Q > 0 \), and along the trajectories of the nonlinear system, we have:

\[
\dot{V}(x) = \dot{x}'P + x'P\dot{x} \\
= [x'A' + f'(x)]Px + x'P[Ax + f(x)] \\
= -x'Qx + 2x'Pf(x) \\
\leq -\lambda_{\min}(Q)\|x\|_2^2 + 2\lambda_{\max}(P)\|f(x)\|_2\|x\|_2 \\
\leq [-\lambda_{\min}(Q) + 2\lambda_{\max}(P)\epsilon]\|x\|_2 \quad \text{whenever} \quad |x_1| \leq \sqrt{\epsilon/2} 
\]

Thus, the function \( V : B_\epsilon \rightarrow \mathbb{R} \) is a Lyapunov function for the nonlinear system in the neighborhood

\[ B_\epsilon = \{x \in \mathbb{R}^2 \|x\|_2 < \sqrt{\epsilon/2}\} \]

for \( \epsilon < \frac{\lambda_{\min}(Q)}{2\lambda_{\max}(P)} \), for any \( Q > 0 \) and corresponding solution \( P \) to the Lyapunov equation; in particular, trajectories starting in this neighborhood will converge asymptotically to the origin. For instance, for \( Q = I \) with \( \lambda_{\min}(Q) = 1 \), we have

\[ P = \frac{1}{4} \begin{bmatrix} 5 & 1 \\ 1 & 1 \end{bmatrix} \]

with \( \lambda_{\max}(P) = 3 + 2\sqrt{5} \) and the corresponding provable region of attraction:

\[ B = \left\{ x \in \mathbb{R}^2 \|x\|_2 < \frac{1}{2} \sqrt{\frac{1}{3 + 2\sqrt{5}}} \right\} \]

Ideally, we would like to choose \( Q > 0 \) to maximize the ratio \( \frac{\lambda_{\min}(Q)}{2\lambda_{\max}(P)} \), which is not an easy problem. What is straightforward though is computing an upper bound for this ratio, which establishes the limitations of this approach in finding a region of attraction. We have:

\[ 2\lambda_{\max}(P)\|A\| = 2\|P\|\|A\| \geq 2\|PA\| \geq \|A'P + PA\| = \| -Q \| \geq \lambda_{\min}(Q) \]

which implies:

\[ \frac{\lambda_{\min}(Q)}{2\lambda_{\max}(P)} \leq \|A\|_2 \]