Problem 1
Solve the following problems in Doyle, Francis and Tannenbaum:

(a) Problem 2.2
(b) Problem 2.4
(c) Problem 2.11
(d) Problem 2.13

Problem 2
Consider the following three systems:

System $S_1$:

\[
\dot{x}_1 = -x_1 + (x_1 + x_2)^3 \\
\dot{x}_2 = -x_2 - (x_1 + x_2)^3 
\]

System $S_2$:

\[
\dot{x}_1 = -x_2 - x_1(x_1^2 + x_2^2) \\
\dot{x}_2 = -x_1 - x_2(x_1^2 + x_2^2) 
\]

System $S_3$:

\[
\dot{x}_1 = x_2|x_1| \\
\dot{x}_2 = -x_1|x_2| 
\]
and the following functions \( V_i : \mathbb{R}^2 \to \mathbb{R}, \)
\[
V_1(x_1, x_2) = x_1^2 + x_2^2 \\
V_2(x_1, x_2) = |x_1| + |x_2| \\
V_3(x_1, x_2) = \max\{ |x_1|, |x_2| \}
\]

(a) What are the equilibrium points of each of these three systems?

(b) We are interested in analyzing stability of the equilibrium point(s) of \( S_1. \) Which of the given functions, if any, is a valid Lyapunov function? Please explain your answers.

**Problem 3**

The objective of this problem is to understand Lyapunov methods for discrete-time LTI systems, by building on what we learned in class for continuous-time systems. Consider a discrete-time LTI system:
\[
x(t + 1) = Ax(t)
\]
and consider the function \( V = x'Px \) where \( P \) is a symmetric positive definite matrix.

(a) Under what conditions is \( V(x) \) a Lyapunov function for this system?

(b) Under what conditions does \( V(x) \) allow us to prove asymptotic stability of the system?

Express your answers in terms of solutions of suitable Lyapunov equations.

**Problem 4**

In this problem, we wish to study the stability of an \( n^{th} \) order LTI system of the form:
\[
\dot{x}(t) = (A + \Delta)x(t)
\]
where \( \Delta \) is a real perturbation. In particular, we wish to find a good lower bound on the size of the smallest perturbation that will destabilize the system. Assume that the given matrix \( A \) is Hurwitz, and define the stability margin of the system as:
\[
\gamma(A) = \min_{\Delta \in \mathbb{R}^{n \times n}} \{ \|\Delta\|_2 | A + \Delta \text{ is not Hurwitz} \}
\]

(a) Consider the Lyapunov equation \( A'P + PA = -Q \) and show that \( \frac{\sigma_{\min}(Q)}{2\sigma_{\max}(P)} \) is a lower bound for \( \gamma(A). \)

(b) An alternative way of computing a lower bound for \( \gamma(A) \) consists of dropping the requirement that \( \Delta \) is real. In particular, show that:
\[
\min_{\Delta \in \mathbb{C}^{n \times n}} \{ \|\Delta\|_2 | A + \Delta \text{ is not Hurwitz} \} = \min_{w \in \mathbb{R}} \sigma_{\min}(A - jwI)
\]

Why does this give us a lower bound on \( \gamma(A)? \)
(c) We can improve on the bound above by using the information that \( \Delta \) is in fact real. Define
\[
A_w = \begin{bmatrix} A & wI \\ -wI & A \end{bmatrix}.
\]
Show that:
\[
\gamma(A) \geq \min_{w \in \mathbb{R}} \sigma_{\min}(A_w).
\]

**Problem 5**

One way of extending the concept of \( p \)-stability to nonlinear systems with an equilibrium point at the origin is by restricting the allowable inputs and assuming the system is initialized at 0, effectively restricting the analysis to a local region around the origin. In this case, a system \( \mathcal{H} \) is said to be locally \( p \)-stable if there exists \( C_\alpha > 0 \) such that:
\[
\|\mathcal{H}(u)\|_p \leq C_\alpha \|u\|_p
\]
for all \( u \) with \( \|u\|_p \leq \delta \). Consider the nonlinear system given by:
\[
\begin{align*}
\dot{x} &= Ax + Bu \\
z &= Cx + Du \\
y &= g(z)
\end{align*}
\]
where \( C \) and \( D \) are assumed to be row matrices (i.e. the system is multi-input single-output) and \( A \) is assumed to be Hurwitz.

(a) When \( g(x) = \cos(x) \), is the system 1-stable? 2-stable? \( \infty \)-stable? Explain your answers.

(b) When \( g(x) = \begin{cases} x & |x| \leq 1 \\ 1 & |x| \geq 1 \end{cases} \), is the system 1-stable? 2-stable? \( \infty \)-stable? Explain your answers.

(c) When \( g(x) = \sin(x) \), is the system 1-stable? 2-stable? \( \infty \)-stable? Explain your answers.