1 Introduction

**Linear System.** Any linear system has a unique solution \( x(t) = e^{At}x_0 \) through each point \( x_0 \in \mathbb{R}^n \) and the solution is defined for all \( t \in \mathbb{R} \).

**Nonlinear Systems.** For nonlinear systems

\[
\dot{x} = f(x)
\]

where \( f : E \to \mathbb{R}^n \) and \( E \) is an open subset of \( \mathbb{R}^n \), the situation is much more subtle.

- For example, the IVP \( \dot{x} = \sqrt{x}, x(0) = 0 \) has 2 different solutions.

- Even \( f \) behaves nicely, \( \dot{x} = x^2, x(0) = 1 \), the solution become unbounded at some finite time.

We will show that

- under certain conditions of \( f \), the nonlinear system has a unique solution through each point \( x_0 \in E \) defined on a maximal interval of existence \( (\alpha, \beta) \subset \mathbb{R} \);

- while it is generally impossible to solve the nonlinear system, a great deal of qualitative information about the local behavior of the solution can be determined via its associated linearized system \( \dot{x} = Df(\bar{x})x \) where \( \bar{x} \) is an equilibrium solution.
2 Existence and Uniqueness

**Definition:** Consider the function $f(t,x)$ with $f : \mathbb{R}^{n+1} \to \mathbb{R}^n$, $|t-t_0| \leq a, x \in E \subset \mathbb{R}^n$. $f(t,x)$ satisfies the Lipschitz condition ($Lipschitz continuous$) w.r.t. $x$ if in $[t_0 - a, t_0 + a] \times E$ we have

$$||f(t,x_1) - f(t,x_2)|| \leq L||x_1 - x_2||$$

with $x_1, x_2 \in E$ and $L$ a constant (Lipschitz constant).

**Remarks:**
- Lipschitz continuous in $x$ implies continuous in $x$.
- Continuous differentiability implies Lipschitz continuity.

**Recall:**
- $f \in C^1(E)$ if (i) $f$ is differentiable for all $x_0 \in E$, i.e., there exists a linear transformation $Df(x_0) \in L(\mathbb{R}^n)$ such that

$$\lim_{|h| \to 0} \frac{|f(x_0 + h) - f(x_0) - Df(x_0)h|}{|h|} = 0$$

and (ii) $Df : E \to L(\mathbb{R}^n)$ is continuous.
- $f \in C^1(E)$ iff $\frac{\partial f}{\partial x_j}$ exist and are continuous on $E$. $Df = \left[\frac{\partial f}{\partial x_j}\right]$.

**Theorem 2.1** Consider the IVP

$$\dot{x} = f(t,x), \quad x(t_0) = x_0$$

with $x \in E \subset \mathbb{R}^n, |t-t_0| \leq a; E = \{x||x - x_0|| \leq d\}$. If

- $f(t,x)$ is continuous in $G = [t_0 - a, t_0 + a] \times E$ and
- $f(t,x)$ is Lipschitz continuous in $x$,

then the IVP has one and only one solution for $|t - t_0| \leq \min(a, d/M)$ with $M = \sup_G ||f||$.

**Theorem 2.2** Let $x_0 \in E \subset \mathbb{R}^n$. If $f \in C^1(E)$, then there exists $b > 0$ such that the IVP has a unique solution $x(t)$ on $[-b,b]$. 


Proof: Based on Picard’s method of successive approximations.

1. Notice that \( x(t) \) is a solution of the IVP iff it is a continuous and satisfies the integral equation
   \[
   x(t) = x_0 + \int_{0}^{t} f(x(s))ds.
   \]

2. The successive approximations to the solution are defined by
   \[
   u_0(t) = x_0 \\
   u_{k+1}(t) = x_0 + \int_{0}^{t} f(u_k(s))ds
   \]

3. For example, consider \( \dot{x} = x, x(0) = 1 \).

4. To show that \( u_k \) converge to a solution. Need to recall:
   - \( C([-a,a]) \) (set of continuous function on \([-a,a]\)) is a complete normed linear space: every Cauchy sequence converges.
   - If \( f \in C^1(E) \), then \( f \) is locally Lipschitz on \( E \).

5. The key step is to show that \( \{u_k\} \) is a Cauchy sequence of continuous functions.

6. Argument for existence:

7. Argument for uniqueness:
**Remarks:** The solution of the IVP will be written as \( x(t) \), \( x(t; x_0) \) or \( x(t; t_0, x_0) \).

- The theorem guarantees the existence of the solution in a neighborhood of \( t = t_0 \), the size of which depends on the supnorm \( M \) of \( f(t, x) \).
- One often can continue the solution outside this neighborhood.

**Theorem 2.3** If \( x \in C^1(M) \) with \( M \) compact, there the system has solution curves defined for all \( t \in \mathbb{R}^n \).

### 3 Dependence on Initial Conditions

**Theorem 3.1** Under the same hypothesis as theorem (2.1), If \( ||\eta|| \leq \epsilon \) then we have

\[
||x_0(t) - x_\epsilon(t)|| \leq \epsilon e^{Lt} \quad \text{on} \quad I
\]

where \( x_0(t), x_\epsilon(t) \) are the solutions to the IVPs \( \dot{x} = f(t, x), x_0(0) = a \) and \( \dot{x} = f(t, x), x_\epsilon(0) = a + \eta \) on interval \( I \) respectively.

**Theorem 3.2** *(Gronwall)* Assume that for \( t_0 \leq t \leq t_0 + a \),

\[
\phi(t) \leq \delta_1 \int_{t_0}^{t} \psi(s) \phi(s) ds + \delta_3
\]

where \( \phi(t) \geq 0, \psi(t) \geq 0, \delta_1 > 0, \delta_3 > 0 \). Then

\[
\phi(t) \leq \delta_3 e^{\delta_1 \int_{t_0}^{t} \psi(s) ds}.
\]

**Theorem 3.3** Under the same hypothesis as above, if

\[
\phi(t) \leq \delta_2(t - t_0) + \delta_1 \int_{t_0}^{t} \phi(s) ds + \delta_3
\]

then

\[
\phi(t) \leq \left( \frac{\delta_2}{\delta_1} + \delta_3 \right) e^{\delta_1 (t - t_0)} - \frac{\delta_2}{\delta_1}.
\]