HIGH RESOLUTION SPECTRA ANALYSIS:
ADVANCES AND APPLICATIONS

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Given time-series data:
\[ \{u_0, u_1, u_2, \ldots, u_{N-1}\} \]

determine the power spectrum of \( y \),
i.e., periodicities and “color”

**Methods:**
- Periodogram, FFT
- Model based (ARMA,...)
- Modern nonlinear (Maximum-entropy, maximum-likelihood,...)
**SYNTHETIC APERTURE RADAR:**

- Antenna
- Satellite flightpath
- Radar pulses
- Beamwidth
- Echo
- Terrain profile
- Beam footprint
- Azimuth
- Range
- Signal at given distance bin across synthetic aperture
- Line by line produces:

![Graphs and images related to SAR applications]
APPLICATIONS: NONINVASIVE TEMPERATURE SENSING

Imaging transducer

Ultrasound echo:

Window corresponding to given depth

⇒

frequency vs. time ⇒ temperature profile
\[ u_k = \int_{-\pi}^{\pi} e^{jk\theta} dv(\theta) \]

\[ \rho(\theta)d\theta \sim E\{dv(\theta)^2\} \quad \text{“energy density across frequencies”} \]

**Covariance statistics & spectral density**

\[ c_k = E\{u_t u_{t+k}\} \]

\[ c_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-jk\theta} \rho(\theta) d\theta \]
Given finite data \( c_0, \ldots, c_N \),

all consistent spectra are given by:

\[
\rho = \Re \left( \frac{A + BQ}{C + DQ} \right) \text{ with } Q \text{ a "free" parameter}
\]
Spectral analysis ⇔ analytic interpolation

- Theory
- Academic examples
- Applications
What is the structure of the state-covariance matrix $\Sigma := E\{xx^*\}$?

What are all spectra consistent with $\Sigma$?

$$
\Sigma = \int_{-\pi}^{\pi} \left( G(e^{j\theta}) \frac{d\mu(\theta)}{2\pi} G(e^{j\theta})^* \right)
$$
\[ G(z^{-1}) = \begin{bmatrix} \frac{1}{z^{-1}} \\ \vdots \\ z^{-n} \end{bmatrix} \]

\[ x_k = \begin{bmatrix} u_k \\ u_{k-1} \\ \vdots \\ u_{k-n} \end{bmatrix} \]

\[ \Sigma = \begin{bmatrix} c_0 & c_1 & \cdots & c_n \\ c_1^* & c_0 & \cdots & c_{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ c_n^* & c_{n-1}^* & \cdots & c_0 \end{bmatrix} \]
\[ \Sigma = \int_{-\pi}^{\pi} \left( G(e^{j\theta}) \frac{d\mu(\theta)}{2\pi} G(e^{j\theta})^* \right) \]

\[ \vdots \]

\[ = BH + H^*B^* + A\Sigma A^* \]

**THM:** With \((A, B)\) controllable pair, \(A\) stable, \(\Sigma \geq 0:\)

\(\Sigma\) is a covariance of \(x_k = Ax_{k-1} + Bu_k\)

\(\iff\)

\(\Sigma = BH + H^*B^* + A\Sigma A^*\) has a solution \(H\)

\(\iff\)

\[ \begin{bmatrix} \Sigma - A\Sigma A^* & B \\ B^* & 0 \end{bmatrix} \]  

\[ \text{rank} \begin{bmatrix} 0 & B \\ B^* & 0 \end{bmatrix} \]

AFOSR meeting on Future Directions in Control
THM: All input spectra consistent with $\Sigma$ are

$$d\mu(\theta) \sim \lim_{r \to 1} \text{Re} \left( F(re^{j\theta}) \right) d\theta$$

where

$$F(\lambda) = F_0(\lambda) + Q(\lambda)V(\lambda) \text{ is positive-real}$$

Data $A, B, \Sigma, H$:

- $F_0(\lambda) = H(I - \lambda A)^{-1} B$,
- $V(\lambda) = D + C\lambda(I - \lambda A)^{-1} B$ all-pass,
- $Q(\lambda)$ free parameter.

$\Rightarrow$ LFT parametrization of all spectra consistent with $\Sigma$
Entropy:

\[ \mathbb{I}(\mu) := \frac{1}{2\pi} \int_{-\pi}^{\pi} \log \det(\hat{\mu}(\theta)) d\theta \]

**THM:** A unique maximizing spectrum is

\[ d\mu_\omega(\theta) := \left( \Phi(e^{j\theta})^{-1} \Omega^{-1} \left( \Phi(e^{j\theta})^{-1} \right)^* \right) d\theta \]

Where:

\[ \Phi(\lambda) := (B^*\Sigma^{-1}B)^{-1} B^*\Sigma^{-1}(I - \lambda A)^{-1}B, \]

\[ \Omega := (B^*\Sigma^{-1}B)^{-1}. \]
If $u_\ell = \text{“white-noise”} + \sum_{k=1}^{m} \sqrt{\rho_k} e^{j\ell \omega_k}$, then

$$\Sigma = \rho_0 \mathbf{I} + \sum_{i=1}^{m} \rho_i G(e^{j\omega_i}) G(e^{j\omega_i})^*$$

\textbf{THM: There exists a unique minimal decomposition of $\Sigma$ corresponding to a sum of sinusoids}
• minimal complexity spectra

• Kullback-Leibler distance & approximation of spectra
\[
\Sigma = \frac{1}{2\pi} \int (G(e^{j\theta}) d\mu(\theta) G(e^{j\theta})^*)
\]
\[ u_k = \nu_k + A_1 \sin(\omega_1 k + \phi_1) + A_2 \sin(\omega_2 k + \phi_2), \quad k = 1, \ldots, n, \]

Noise, sinusoid 1, sinusoid 2, and their sum

\[ \omega_2 - \omega_1 < \frac{2\pi}{n} = \text{Fourier uncertainty bound} \]
Fractal spectrum
limits to resolution?

periodogram
FRACTAL SPECTRUM:
with ME interpolants

Maximum entropy spectra
- - - Focusing filter
SAR IMAGING:
FILTER FOCUSING
fft

high resolution
Non-invasive Ultrasound Temperature Sensing

Imaging transducer

Heating transducer

Ultrasound echo:

Periodogram analysis vs. high resolution methods

Comparison with thermocouple
(thermocouple, periodogram, high resolution)

• Collaboration: E. Ebbini
SUMMARY

- generalized statistics ~ analytic interpolation
- high resolution, applications

QUESTIONS AND ON-GOING RESEARCH PROGRAM:

- ¿how can we quantify resolution?
- tradeoffs between variance and resolution seeking an “$H_{\infty}$-like paradigm”
- spacio-temporal dynamics and non-uniform arrays
- applications: SAR, medical imaging, polarimetry

Matlab code and references at:
http://www.ece.umn.edu/users/georgiou