1. Consider the following vector field in $\mathbb{R}^3$:

\[
\begin{align*}
\dot{x} &= -x + y + f \\
\dot{y} &= -y + g \\
\dot{z} &= z
\end{align*}
\]

where $f(x, y, z) = -(x + \frac{1}{2}y)^3$ and $g(x, y, z) = -(y + \frac{1}{2}x)^3$.

(a) Compute the linearized system at the origin and write it in the form

\[\dot{x} = Ax\]

for a suitable $3 \times 3$ matrix $A$ and where $x$ is the vector with components $(x, y, z)$.

(b) Sketch the phase portrait of this linear system.

(c) To what extent is the phase portrait of the nonlinear system similar to that of the linear system in a neighborhood of the origin?

(d) Consider the function

\[V(x, y, z) = \frac{1}{2} \left[ x^2 + y^2 + xy \right]\]

Compute its time derivative along the flow of the given vector field.

(e) Show that the plane $z = 0$ is invariant.

(f) Is the origin globally attracting within the plane $z = 0$?

(g) Describe the invariant manifolds of the origin for this system.

(h) Can this vector field have any periodic orbits?
2. Consider the planar double pendulum shown in the figure. Use the notation in the figure and assume that the pendulum swings in the vertical plane and ignore frictional forces.

\[ l_i = \text{pendulum lengths} \]
\[ m_i = \text{pendulum bob masses} \]
\[ g = \text{gravitational acceleration} \]
\[ \theta_i = \text{angle of pendula from the downward vertical} \]

(a) Find the Lagrangian for the system.
(b) Compute the Euler-Lagrange equations.
(c) Is the downward equilibrium solution (with \( \theta_1 = \theta_2 = 0 \)) (linearly or nonlinearly) stable?
(d) What about the upward equilibrium solution (with \( \theta_1 = \theta_2 = \pi \))?
(e) Does the system have any periodic orbits?