

Using Lyapunov Vectors and Dichotomy to Solve Hyper-Sensitive Optimal Control Problems

Ufuk Topcu and Kenneth D. Mease

Abstract—The dichotomic basis method for solving completely hyper-sensitive optimal control problems is modified by using Lyapunov exponents and vectors. It is shown that the asymptotic Lyapunov vectors form dichotomic transformations that decouple the unstable dynamics from the stable dynamics. For numerical implementation, finite-time Lyapunov vectors are used to approximate the asymptotic Lyapunov vectors and to construct an approximate dichotomic basis. A re-initialization process is introduced to decrease the error accumulation. The new basis identifies the stable and unstable directions more accurately than the eigenvectors of the Jacobian matrix.

I. INTRODUCTION

An optimal control problem and its associated Hamiltonian boundary value problem (HBVP) are called completely hyper-sensitive, if the time interval of interest is very long relative to the minimum rate of contraction and expansion of the Hamiltonian system in the neighborhood of the solution to the HBVP [1], [2], [3]. The ill-conditioning due to the sensitive dependence on the unknown initial conditions complicates the solution of hyper-sensitive HBVPs. Achieving a specified accuracy on the terminal boundary conditions requires the initial boundary conditions in the fast expanding directions to be satisfied much more accurately potentially exceeding the available precision. Although the indirect shooting method can be modified using multiple shooting nodes [4], denser in the boundary-layers, to handle the hyper-sensitivity, it does not provide any information regarding the geometric structure of the solution. Such information could provide insight regarding the optimal solution and aid the development of a simpler approximate solution method that produces a near-optimal solution.

For completely hyper-sensitive optimal control problems, an indirect solution method based on the underlying geometric structure of the trajectories of the Hamiltonian system in the state-costate space has been introduced [2], [3]. The method uses a dichotomic transformation to decouple the contracting and expanding behavior. The solution to the HBVP is constructed by forward integration of the contracting (stable) part of the dynamics and backward integration of the expanding (unstable) part of the dynamics. The use of dichotomic transformations for the solution of linear boundary value problems is well known in the numerical analysis literature [5]. Wilde and Kokotovic applied Riccati equation based dichotomic transformations to linear fixed

end-point optimal control problems and computed the solution as the composition of two initial value problems [6]. This methodology was extended to nonlinear problems in [7]. However, the applicability of this method is limited because the system representation is required to be in a special input-affine form. Chow used dichotomic transformations for nonlinear two-timescale HBVPs in standard singularly perturbed form with linear boundary-layer dynamics [8].

Guided by the geometric characterization of two-timescale systems by Fenichel [9] and the stable and unstable sub-bundles of Sacker and Sell [10], Rao and Mease extended the use of dichotomic transformations to nonlinear HBVPs [2], [3]. They used the eigenvectors of the Jacobian of the Hamiltonian vector field as the basis vectors to construct approximate dichotomic transformations. It is well known that the eigenvectors do not provide the correct structure of the linear time-varying (LTV) system which results when the nonlinear Hamiltonian system is linearized around a non-stationary reference trajectory. To overcome this, the initial and terminal boundary-layers of the composite solution were computed by successive approximations inspired by the computational singular perturbation methodology [11].

In [12], [13], it is shown that the Lyapunov vectors give accurate information on the geometry of the linearized time-varying dynamics of the nonlinear systems. In this paper, we use appropriate Lyapunov vectors to construct a dichotomic basis. Our aim is to develop a methodology for partially hypersensitive HBVPs, but in this paper we only take the first step of considering the completely hypersensitive case. Although other approaches that take advantage of the well-developed theory of the dynamics near an equilibrium point are applicable for the completely hypersensitive case, our approach has the advantage of being applicable to the partially hypersensitive case, where “partially” denotes that in addition to fast contracting and expanding behavior, there is nontrivial slower behavior. The paper is organized as follows: In section II, the problem is introduced and the geometric structure of its solution is explained on a sample problem. The dichotomic basis method is described in section III. We review the Lyapunov exponents and vectors and show that appropriate Lyapunov vectors form a dichotomic basis in section IV. In section V is devoted to the approximate dichotomic basis from finite-time Lyapunov vectors. The sample problem is solved using the new method in section V.

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U. Topcu is with the Department of Mechanical Engineering, University of California, Berkeley, CA 94720, USA. utopcu@berkeley.edu

K.D. Mease is with Faculty of Mechanical and Aerospace Engineering, University of California, Irvine, CA 92697, USA. kmease@uci.edu

II. HBVP AND ASSOCIATED GEOMETRIC STRUCTURE

A. Hamiltonian boundary value problem

Consider the optimal control problem: find a piecewise continuous control $u: \mathbb{R} \rightarrow \mathbb{R}^m$ that minimizes the scalar cost

$$\mathcal{J} = \int_0^{t_f} \mathcal{C}(s, u) dt \quad (1)$$

subject to the dynamic constraints $\dot{s} = f(s, u)$, and boundary conditions

$$s(0) = s_0, \quad s(t_f) = s_f, \quad (2)$$

where $s: \mathbb{R} \rightarrow \mathbb{R}^n$ is the state and the final time t_f is given. The first-order necessary conditions for optimality lead to a HBVP with the dynamics $\dot{s} = H_\lambda(s, \lambda)^T$, $\dot{\lambda} = -H_s(s, \lambda)^T$, and the boundary conditions (2), where $\lambda: \mathbb{R} \rightarrow \mathbb{R}^n$ is the costate, $H(s, \lambda) = \mathcal{C}(s, u(s, \lambda)) + \lambda^T f(s, u(s, \lambda))$ is the Hamiltonian evaluated for the optimal control $u(s, \lambda) = \arg \min_u H(s, \lambda, u)$, and H_λ and H_s are partial derivatives with respect to the subscripted variable. We assume that H is a smooth function of s and λ . To simplify notation, let $x = (s^T \lambda^T)^T$ and express the state-costate dynamics as

$$\dot{x} = f(x) = (H_\lambda(x) \quad -H_s(x))^T. \quad (3)$$

The n -dimensional s -space and the $2n$ -dimensional (s, λ) -space are called the state space and the phase space, respectively.

B. Geometric structure of the solution to a completely hyper-sensitive HBVP

In the phase space, the solution trajectory to a completely hyper-sensitive HBVP [2], [3] is intimately related to the geometric structure of a saddle-type equilibrium point [14]. The complete solution trajectory begins slightly off the stable manifold of the equilibrium point and follows it quickly towards the equilibrium point. The trajectory then progresses slowly in the vicinity of the equilibrium point. Near the final time, the trajectory quickly follows the unstable manifold of the equilibrium point to the final condition slightly off the unstable manifold. For the fixed boundary conditions on the state, as the final time increases, the initial and terminal boundary-layer segments of the trajectory lie closer and closer to the stable and unstable manifolds of the equilibrium point, respectively [2], [3].

Example 2.1: We consider the optimal control problem with $\mathcal{C}(s, u) = s^2 + u^2$, state dynamics $\dot{s} = -s^3 + u$, and the boundary constraints $s(0) = 1$ and $s(t_f) = 1.1$, where $s(t), u(t) \in \mathbb{R}$. Applying the first-order conditions for optimality leads to the HBVP $[\dot{s}, \dot{\lambda}]^T = [-s^3 - \lambda/2, -2s + 3s^2\lambda]^T$. The solutions to the problem for the final times $t_f = 1, 3, 5, 10, 20$ are shown in Figure 1. Note that the solution trajectories approach the stable and unstable manifolds of the equilibrium at the origin and the initial and terminal boundary-layers for $t_f = 10, 20$ are indistinguishable from the stable and unstable manifolds. \diamond

Based on the geometric structure of the phase space around the solution of the hyper-sensitive HBVPs, an approximate

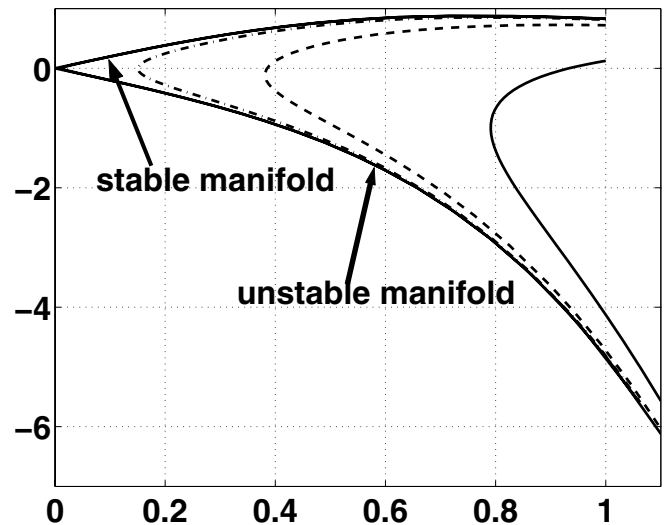


Fig. 1. Stable and unstable manifolds and trajectories with $t_f = 1, 3, 5, 10, 20$. Trajectories approach the stable and unstable manifolds as t_f increases.

composite solution can be given for sufficiently large final times as

$$\hat{x}(t) = \begin{cases} x_s(t), & 0 \leq t \leq t_{ibl} \\ x_{eq}(t), & t_{ibl} < t < t_f - t_{fbl} \\ x_u(t), & t_f - t_{fbl} \leq t \leq t_f \end{cases}, \quad (4)$$

where $x_{eq}(t) = x_{eq}$ is the equilibrium solution, $x_s(t)$ is the solution to (3) with the initial condition $x_s(0) = (s_0, \lambda_0)$ and λ_0 is chosen such that $x_s(0)$ is on the stable manifold of x_{eq} , and $x_u(t)$ is the solution to (3) with final condition $x_u(t_f) = (s_f^T \lambda_f^T)^T$ and λ_f chosen such that $x_u(t_f)$ is on the unstable manifold of x_{eq} . The duration of the initial boundary-layer is t_{ibl} , and the duration of the final boundary-layer is t_{fbl} .

III. SOLUTION VIA DICHOTOMIC BASIS

We will characterize points on the stable manifold by decomposing the vector field f into stable and unstable components, $f(x) = f_s(x) + f_u(x)$, and using the property $f_u(x_0) = 0$ for x_0 on the stable manifold. The stable/unstable decomposition of f can be achieved by representing it in a dichotomic basis.

The solution to (3) for the initial condition $x(0) = x_0$ is denoted by $x(t) = \phi(t, x_0)$, where ϕ is the transition map or flow and satisfies (3), i.e., $\dot{\phi} = f(\phi(t, x_0))$ and $\phi(0, x_0) = x_0$. The tangent space at a point $x \in \mathbb{R}^{2n}$ is denoted by $T_x \mathbb{R}^{2n}$. The tangent bundle $T\mathbb{R}^{2n}$ is defined as the union of the tangent spaces over \mathbb{R}^{2n} and (x, v) is a point in the tangent bundle, with v the tangent vector and x the base point. A tangent vector evolves according to linearized dynamics associated with (3)

$$\dot{v} = J(x)v, \quad (5)$$

where $J = \partial f / \partial x$ is the Jacobian matrix and v can be thought as a small perturbation from a reference trajectory. The coordinates for v correspond to a tangent space frame whose axes are parallel to the phase space axes, but with

origin at x . (3) and (5) are a coupled system of differential equations, because $J(x)$ depends on x . An initial point (x, v) is mapped in time t to the point $(\phi(t, x), \Phi(t, x)v)$, where Φ is the transition matrix for the linearized dynamics and satisfies (5) with the initial condition $\Phi(0, x) = I_{2n}$ (I_d is the identity matrix in $\mathbb{R}^{d \times d}$). Note that the arguments of the transition map and the transition matrix are the propagation duration t and initial point x .

Consider a coordinate change $(x, v) \mapsto (x, w)$ on $T\mathbb{R}^{2n}$ with $v = D(x)w$, where $D(x) \in \mathbb{R}^{2n \times 2n}$ is a nonsingular and continuously differentiable function of x on \mathbb{R}^{2n} . The columns of $D(x)$ are basis vectors for $T_x\mathbb{R}^{2n}$ for each $x \in \mathbb{R}^{2n}$. In terms of w , the variational equation (5) becomes

$$\dot{w} = [D(x)^{-1}J(x)D(x) - D(x)^{-1}\dot{D}(x)]w = \Lambda(x)w, \quad (6)$$

where $\dot{D}(x)$ is the time derivative of each element of $D(x)$, and $\Lambda(x)$ denotes the transformed system matrix.

Definition 3.1: A basis D is called dichotomic, if

- 1) The corresponding system matrix Λ is block-diagonal of the form

$$\dot{w} = \Lambda(x)w = \begin{bmatrix} \Lambda_s(x) & 0 \\ 0 & \Lambda_u(x) \end{bmatrix} w \quad (7)$$

where the matrices $\Lambda_s, \Lambda_u \in \mathbb{R}^{n \times n}$.

- 2) The transition matrices $\Phi_s(t, x)$ and $\Phi_u(t, x)$ satisfying $\dot{\Phi}_s(t, x) = \Lambda_s(x)\Phi_s(t, x)$ with $\Phi_s(0, x) = I_n$ and $\dot{\Phi}_u(t, x) = \Lambda_u(x)\Phi_u(t, x)$ with $\Phi_u(0, x) = I_n$ also satisfy

$$\begin{aligned} \|\Phi_s(t, x)\| &\leq Ke^{-\beta t}, \quad t \geq 0, \\ \|\Phi_u(t, x)\| &\leq Ke^{\beta t}, \quad t \leq 0, \end{aligned} \quad (8)$$

where β and K are positive constants and K is of moderate size [5].

Note that the stable and unstable subsystems are decoupled in the transformed coordinates. A dichotomic basis D can be partitioned as $D(x) = [D_s(x) \ D_u(x)]$, where $D_s(x), D_u(x) \in \mathbb{R}^{2n \times n}$. Correspondingly, the new coordinate vector is decomposed as $w = (w_s^T \ w_u^T)^T$, where $w_s, w_u \in \mathbb{R}^n$. At each phase point x , the vector $f(x)$ can be expressed in terms of the dichotomic basis vectors as

$$\dot{x} = \underbrace{D_s(x)w_s(x)}_{f_s(x)} + \underbrace{D_u(x)w_u(x)}_{f_u(x)}, \quad (9)$$

where the coordinates w_s and w_u are determined by $w_s(x) = D_s^\dagger(x)f(x)$, and $w_u(x) = D_u^\dagger(x)f(x)$. $D_s^\dagger(x) \in \mathbb{R}^{n \times 2n}$ and $D_u^\dagger(x) \in \mathbb{R}^{n \times 2n}$ are the first and second n rows of $D(x)^{-1}$, respectively. An alternative to integrating (3) to compute the extremal solution for a given initial condition $x(0) = x_0 = (s_0^T \ \lambda_0^T)^T$ is to integrate the system (7) and (9) with the initial conditions $x(0) = x_0$, $w_s(0) = D_s^\dagger(x_0)f(x_0)$, $w_u(0) = D_u^\dagger(x_0)f(x_0)$. This alternative makes clear that for a given initial condition on the state, $s(0) = s_0$, the unstable component of the Hamiltonian vector field can be suppressed at $t = 0$ by choosing $\lambda(0) = \lambda_0$ such that $w_u(s_0, \lambda_0) = 0$; and that the unstable component will remain suppressed, due to the lack of coupling from w_s to w_u . The initial boundary-layer solution can be computed by integrating (7) and (9) to

ensure that the trajectory remains on the stable manifold. The integration is continued until the trajectory is close enough to the equilibrium, such that $\|x(t_{ibl}) - x_{eq}\| \leq \varepsilon$ for a specified tolerance ε , to connect it to the equilibrium solution. Next we describe how λ_0 can be chosen so that $w_u(s_0, \lambda_0) = 0$ is satisfied.

Let $\mathcal{V}_s(x) = \mathcal{R}(D_s(x))$ and $\mathcal{V}_u(x) = \mathcal{R}(D_u(x))$ denote the stable and unstable subspaces of the tangent space $T_x\mathbb{R}^{2n}$ where $\mathcal{R}(\cdot)$ denotes the range space. For each $x \in \mathcal{S}$, where \mathcal{S} is the stable manifold, $\mathcal{V}_s(x)$ coincides with $T_x\mathcal{S}$, the tangent space of \mathcal{S} at the point x . \mathcal{S} is an invariant manifold [14]. A requirement for invariance is that $f(x) \in T_x\mathcal{S}$ for all $x \in \mathcal{S}$, which means that the trajectory through x is tangent to \mathcal{S} and will thus stay on \mathcal{S} . Hence, $f(x) \in \mathcal{V}_s(x)$ for all $x \in \mathcal{S}$, i.e., $f(x_0) \in \mathcal{V}_s(x_0)$ implies that $w_u(s_0, \lambda_0) = 0$.

For the terminal boundary-layer, λ_f is chosen such that $w_s(s_f, \lambda_f) = 0$ to ensure that the terminal boundary-layer segment is on the unstable manifold.

IV. EXACT DICHOTOMIC BASIS

Lyapunov exponents indicate the average exponential rates for solutions $v(t)$ of $\dot{v} = J(x)v$ along trajectories of the nonlinear system $\dot{x} = f(x)$. Lyapunov vectors indicate the directions associated with each average exponential rates. Lyapunov exponents [15] are the appropriate measures for defining stable and unstable motion in the linearized dynamics $\dot{v} = J(x)v$ when $J(x)$ is not constant.

Hereafter we use the notation μ^+ and l^+ for the Lyapunov exponents and vectors. This is to emphasize that the Lyapunov exponents and vectors we use are the forward time Lyapunov exponents and vectors. From the forward-time Lyapunov vectors, we can construct $D_s(x)$. Since this is sufficient to compute the initial boundary-layer approximation, we only discuss the construction of D_s . With the ‘‘partial’’ dichotomic transformation $D = [D_s \ D_m]$, with D_m completing the basis but $D_m \neq D_u$, we obtain an upper block triangular transformed system in place of the form in (7). We can use backward Lyapunov exponents and vectors to construct D_u which is needed for the terminal boundary-layer. The development is parallel to that for the forward time case.

A. Lyapunov exponents and vectors

Consider a compact invariant set $\mathcal{Y} \subset \mathbb{R}^{2n}$, where $J(x)$ is bounded and smooth enough so that solutions for $\dot{v} = J(x)v$ exist for all times. Lyapunov exponents at a Lyapunov regular point $x \in \mathcal{Y}$ can be defined as [16]

$$\mu^+(x, v) = \lim_{T \rightarrow \infty} \frac{1}{T} \ln \frac{\|\Phi(T, x)v\|}{\|v\|} \quad (10)$$

for any nonzero $v \in T_x\mathbb{R}^{2n}$. For $v = 0$, $\mu^+(x, v) = -\infty$ for each $x \in \mathcal{Y}$. The Lyapunov exponents can take on at most $2n$ distinct values [16]. For simplicity, we assume that μ^+ takes on exactly $2n$ distinct values, and these distinct values are ordered as $\mu_1^+ < \dots < \mu_{2n}^+$. For a given trajectory in \mathcal{Y} the Lyapunov exponents are independent of the starting point.

If a collection of $r \leq 2n$ linear subspaces of $T_x \mathbb{R}^{2n}$ can be ordered such that $\Lambda_1(x) \subset \Lambda_2(x) \subset \dots \subset \Lambda_r(x) = T_x \mathbb{R}^{2n}$, then this collection of nested proper subspaces defines a filtration of $T_x \mathbb{R}^{2n}$. The linear subspaces $\mathcal{L}_i^+(x)$, defined as $\mathcal{L}_i^+(x) = \{v \in T_x \mathbb{R}^{2n} : \mu^+(x, v) \leq \mu_i^+\}$, for $i = 1, \dots, 2n$, with $\mathcal{L}_0^+ = \{0\}$, form a filtration of $T_x \mathbb{R}^{2n}$ as $\{0\} = \mathcal{L}_0^+ \subset \mathcal{L}_1^+(x) \subset \mathcal{L}_2^+(x) \subset \dots \subset \mathcal{L}_{2n}^+(x) = T_x \mathbb{R}^{2n}$. The distributions $\mathcal{L}_i^+(x) = \text{span}\{l_1^+(x), \dots, l_i^+(x)\}$ are each invariant with respect to the linear flow associated with (5) [16], [17].

An orthonormal basis for $\mathcal{L}_i^+(x)$ can be formed in the following manner [15]: Let the one dimensional subspace $\mathcal{L}_1^+(x)$ be spanned by the unit vector $l_1^+(x)$, called the Lyapunov vector corresponding to μ_1^+ at x . Then, define another unit vector $l_2^+(x)$, called the Lyapunov vector corresponding to μ_2^+ at x , orthogonal to $l_1^+(x)$ such that $\mathcal{L}_2^+(x) = \text{span}\{l_1^+(x), l_2^+(x)\}$. Continuing this procedure creates a new representation of $\mathcal{L}_i^+(x) = \text{span}\{l_1^+(x), \dots, l_i^+(x)\}$, where $l_i^+(x)$ is the Lyapunov vector corresponding to μ_i^+ at x .

The linear flow does not preserve the mutual orthogonality of the Lyapunov vectors. The differential equations, that can be utilized to propagate any of the Lyapunov vectors from $\phi(0, x)$ to other points, $\phi(t, x)$, along the trajectory and preserve the orthonormality and invariance properties, are [18]

$$\begin{aligned} \dot{l}_i^+(x) = & J(x)l_i^+(x) - (l_i^+(x)^T J(x)l_i^+(x))l_i^+(x) \\ & - \sum_{j=1}^{i-1} (l_i^+(x)^T J(x)l_j^+(x) + l_j^+(x)^T J(x)l_i^+(x))l_j^+(x). \end{aligned} \quad (11)$$

These equations are derived by performing a continuous Gram-Schmidt re-orthonormalization starting with the Lyapunov vector corresponding to the smallest exponent, namely l_1^+ .

B. Lyapunov vectors as dichotomic basis

Consider the Lyapunov vector basis $D = [D_s \ D_m]$ where $D_s(x) = [l_1^+(x) \dots l_n^+(x)]$, and $D_m(x) = [l_{n+1}^+(x) \dots l_{2n}^+(x)]$. Using the differential equation for $l_i^+(x)$ in (11), $\dot{D}(x)$ can be written as $\dot{D}(x) = [\dot{l}_1^+(x) \dots \dot{l}_{2n}^+(x)] = J(x)D(x) - D(x)R(x)$, where $R(x) \in \mathbb{R}^{2n \times 2n}$ is upper-triangular and can be written as

$$R(x) = \begin{cases} l_i^+(x)^T J(x)l_j^+(x) + l_j^+(x)^T J(x)l_i^+(x), & \text{if } i < j \\ l_i^+(x)^T J(x)l_j^+(x), & \text{if } i = j \\ 0, & \text{if } i > j \end{cases}. \quad (12)$$

Substituting $\dot{D}(x)$ into (6) gives

$$\begin{aligned} \dot{w} &= [D^{-1}JD - D^{-1}\dot{D}]w \\ &= [D^{-1}JD - D^{-1}JD + D^{-1}DR]w = R w, \end{aligned} \quad (13)$$

in upper block triangular form. All Lyapunov vectors have, by definition, unit length; thus D is bounded. Because J is bounded, \dot{D} is bounded. Since D is orthonormal, $D^{-1} = D^T$. Therefore, D^{-1} is bounded. Thus, D is a Lyapunov transformation and preserves the Lyapunov exponents [19], i.e., $\dot{v} = J(x)v$ and $\dot{w} = R(x)w$ have the same Lyapunov exponents. The Lyapunov vector $l_i^+(x)$ is represented by $(0, \dots, 0, 1, 0, \dots, 0)^T$, where i -th entry is 1, in w -coordinates.

So the Lyapunov exponents of the upper diagonal block are the smallest n Lyapunov exponents of the original system. By taking $\beta = \max \mu_n^+(x)$ over Y , it follows that

$$v \in \mathcal{R}(D_s(x)) \Rightarrow \|\Phi(t, x)v\| \leq K e^{-\beta t} \|v\|, \quad t > 0, \quad \forall x \in \mathcal{X}. \quad (14)$$

for some moderate $K > 0$.

Thus we have shown that a partial dichotomic basis can be formed from the asymptotic Lyapunov vectors. However, many optimal control problems are posed on compact sets that are not invariant, violating a key hypothesis on which the above results are based. When the set is not invariant, the timescales and the associated geometric structure characteristics of the flow on the subset must be computable on finite-time trajectory segments. In the next section, a method for extracting timescale information and its associated tangent space structure on a compact non-invariant subset in the presence of a timescale gap, requiring only finite-time integration, is presented. Details on the finite-time approximations and another application of finite-time Lyapunov exponents and vectors for determining the slow manifold in two-timescale dynamics can be found in [13].

V. APPROXIMATE DICHOTOMIC BASIS

A. Finite-time Lyapunov exponents and vectors

We now consider the system behavior on a (not necessarily invariant) compact subset $\mathcal{X} \subset \mathbb{R}^{2n}$. The finite-time Lyapunov exponents are given by

$$\mu^+(T, x, v) = \frac{1}{T} \ln \frac{\|\Phi(T, x)v\|}{\|v\|} \quad (15)$$

for propagation time $T > 0$. For $v = 0$, define $\mu^+(T, x, 0) = -\infty$. Discrete Lyapunov spectra, for each (T, x) , can be defined in terms of the extremal values of the exponents. These extremal values are the exponents for the unit vectors in $T_x \mathbb{R}^{2n}$ that map to the principal axes of an ellipsoid in $T_{\phi(T, x)} \mathbb{R}^{2n}$. There are at most $2n$ distinct extremal values for each (T, x) . We assume that the extremal exponents are distinct and ordered as $\mu_1^+(T, x) < \dots < \mu_{2n}^+(T, x)$ for each (T, x) . This assumption is used in the hypothesis of Theorem (5.2). In this case we say that the Lyapunov spectrum is *non-degenerate with uniformly bounded gaps*.

The extremal exponents can be computed from the singular value decomposition (SVD) of $\Phi(T, x) = N^+(T, \phi(T, x))\Sigma^+(T, x)L^+(T, x)^T$, $\Sigma^+(T, x) = \text{diag}(e^{\mu_1^+(T, x)T}, \dots, e^{\mu_{2n}^+(T, x)T})$. The column vectors of $L^+(T, x)$ denoted $l_i^+(T, x), i = 1, \dots, 2n$, are the finite-time Lyapunov vectors and the column vectors of $N^+(T, \phi(T, x))$ are denoted $n_i^+(T, \phi(T, x)), i = 1, \dots, 2n$ are the principal axes of the ellipsoid in $T_{\phi(T, x)} \mathbb{R}^{2n}$. From the properties of the SVD, it follows that $\Phi(T, x)l_i^+(T, x) = e^{\mu_i^+(T, x)T}n_i^+(T, \phi(T, x))$. Although the asymptotic Lyapunov exponents are constant along the trajectories and the asymptotic Lyapunov vectors are only functions of the base point $x \in \mathcal{X}$, the finite-time Lyapunov exponents and vectors depend on the base point and the averaging time. The finite-time Lyapunov vectors will be

used to define subspaces in $T_x\mathbb{R}^{2n}$ with different exponential rates.

The column vectors of $L^+(T, x)$ provide an orthonormal basis for $T_x\mathbb{R}^{2n}$. The subspaces $\mathcal{L}_i^+(T, x)$, for $i = 1, \dots, 2n$, are defined by the Lyapunov vectors as $\mathcal{L}_i^+(T, x) = \text{span}\{l_1^+(T, x), \dots, l_i^+(T, x)\}$ and form the filtration $\{0\} = \mathcal{L}_0^+ \subset \mathcal{L}_1^+(T, x) \subset \mathcal{L}_2^+(T, x) \subset \dots \subset \mathcal{L}_{2n}^+(T, x) = T_x\mathbb{R}^{2n}$. Note that for each $v \in \mathcal{L}_i^+(T, x)$, $\mu^+(T, x, v) \leq \mu_i^+(T, x)$.

The feasibility of determining timescale structure and reducing the coupling between the stable and unstable dynamics depends on whether $\mathcal{L}_n^+(T, x)$ converges to a sufficient degree, as the averaging time increases, within the available range of averaging times.

Definition 5.1: [13] For a specified $T_0 > 0$, the relative spectral gap $\Delta\mu_j^+(x)$ between neighboring finite-time exponents $\mu_j^+(T, x)$ and $\mu_{j+1}^+(T, x)$, for some $j \in \{1, 2, \dots, 2n - 1\}$, is $\Delta\mu_j^+(x) = \inf_{T > T_0} (\mu_{j+1}^+(T, x) - \mu_j^+(T, x))$.

T_0 is introduced to eliminate any initial transient period that is not representative of the subsequent behavior and should be small relative to the available averaging time. Although our main interest is in non-invariant subsets of the phase space, consider for the moment the following hypothetical setting in which \mathcal{X} is contained in a set \mathcal{Y} that is invariant under the dynamics in (3) and the uniform timescale properties of \mathcal{X} are satisfied on all of \mathcal{Y} . In this hypothetical setting, we can make a connection between the finite-time tangent space structure and the analogous structure for asymptotic behavior. First we note that if $x \in \mathcal{Y}$ is a regular point, then the Lyapunov exponents converge, $\lim_{T \rightarrow \infty} \mu^+(T, x, v) = \mu^+(x, v)$. The following theorem describes how the finite-time Lyapunov subspaces evolve toward their infinite-time limits.

Theorem 5.2: [13] Consider the dynamical system $\dot{x} = f(x)$ on a compact invariant set \mathcal{Y} of the phase space \mathbb{R}^{2n} . At a point $x \in \mathcal{Y}$ for which there exists $T_0 > 0$ such that the Lyapunov spectrum is non-degenerate with uniformly bounded gaps for $T > T_0$, the subspace $\mathcal{L}_n^+(T, x)$ approaches a fixed subspace $\mathcal{L}_n^+(x)$ at an exponential rate characterized, for every sufficiently small $\Delta T > 0$, by the existence of a constant $C > 0$ such that

$$\text{dist}(\mathcal{L}_n^+(T, x), \mathcal{L}_n^+(T + \Delta T, x)) \leq C e^{-\Delta\mu_n^+(x)T} \quad (16)$$

for $T > T_0$, where $\Delta\mu_n^+(x)$ is the relative spectral gap defined above and $\text{dist}(\mathcal{L}_n^+(T, x), \mathcal{L}_n^+(T + \Delta T, x)) := \|L_n^+(T, x)^T L_n^+(T + \Delta T, x)\|$, and $L_n^+(T + \Delta T, x) := [l_{n+1}^+(T + \Delta T, x) \dots l_{2n}^+(T + \Delta T, x)]$.

B. Approximate dichotomic basis:

We define the finite-time approximation of the stable subspace $\mathcal{V}_s(x)$ by $\hat{\mathcal{V}}_s(T, x) = \mathcal{L}_n^+(T, x)$. The accuracy with which $\hat{\mathcal{V}}_s(T, x)$ approximates $\mathcal{V}_s(x)$ depends on the size of the relative spectral gap relative to the available averaging time.

For the completely hyper-sensitive HBVPs, the finite-time Lyapunov spectrum is symmetric about zero for each (T, x) . There exists $\alpha(x) > 0$ such that

$$\mu_n^+(T, x) \leq -\alpha(x) \text{ and } \alpha(x) \leq \mu_{n+1}^+(T, x) \quad (17)$$

for all T and each $x \in \mathcal{X}$. Then, the relative spectral gap can be taken as $\Delta\mu_n^+(x) = 2 \max \alpha(x)$ such that (17) holds. Regardless of how large this gap is, the subspaces $\mathcal{L}_n^+(T, x)$ will converge to $\mathcal{L}_n^+(x)$ in \mathcal{Y} as $T \rightarrow \infty$. However, we require this gap to be large enough so that the finite-time subspaces computed by averaging only along the trajectory segments in \mathcal{X} approximate the asymptotic subspaces with sufficient accuracy. The condition we are looking for is

$$\frac{(\Delta\mu_n^+(x))^{-1}}{T_c - T_0} \leq \varepsilon_{\text{gap}} \quad \text{for all } x \in \mathcal{X}, \quad (18)$$

where T_c , the largest averaging time. In practice, ε can be chosen in the range $5^{-1} - 3^{-1}$.

Now, let $\hat{D}(T, x) \in \mathbb{R}^{2n \times 2n}$ be the matrix whose columns are given as $\hat{D}(T, x) = [l_1^+(T, x) \dots l_{2n}^+(T, x)]$. Since the columns of $\hat{D}(T, x)$ do not form a partial dichotomic basis $\hat{\Lambda}(x) = \hat{D}(T, x)^{-1} J(x) \hat{D}(T, x) - \hat{D}(T, x)^{-1} \hat{D}(T, x)$ will not be in block-triangular form. However, (17) implies that the span of the column vectors of $L_n^+(T, x)$ will be close to its hypothetical infinite time limit. In addition, when the available averaging time is long enough, i.e., condition (18) is satisfied, $\mathcal{R}(L_n^+(T, x))$ will approximate $\mathcal{V}_s(x)$ accurately. The accuracy will increase as the ratio $\frac{\Delta\mu(x)^{-1}}{T_c - T_0}$ decreases. The finite-time Lyapunov vectors can then be used to construct a transformation to approximately decouple the stable and unstable modes. A set of linearly independent vectors that approximately decouple the linearized dynamics is called ‘‘approximate dichotomic basis’’ in [2], [3]. We use the finite-time Lyapunov vectors as an approximate dichotomic basis.

The key benefit of the composite approximation approach is that error growth, an inherent feature of numerically integrating a Hamiltonian system, is only taking place over the time interval t_{ibl} , which is much smaller than t_f . Another way to attenuate the error growth is re-initialization of the propagation along the trajectory. Let $[0, t_{ibl}]$ be subdivided into intervals $[t_{k-1}, t_k]$, $k = 1, \dots, r$, where $t_0 = 0$ and $t_r = t_{ibl}$. By using $\hat{D}_u^+(T, x(t_{k-1})) f(x(t_{k-1})) = \hat{w}_u(0) = 0$, $k = 1, \dots, r$, the new costate corresponding to the current state can be found. This changes the costate such that $(s(t_{k-1}), \lambda(t_{k-1}))$ is closer to the stable manifold. At the expense of re-calculating the Lyapunov vectors and solving the algebraic equation $\hat{w}_u(0) = 0$, re-initialization decreases the error accumulation and increases the accuracy of the method.

VI. EXAMPLE

Example (2.1) is solved using the method introduced in the previous sections. The initial boundary-layer is computed using the dichotomic basis method with finite-time Lyapunov vectors. The region $\mathcal{X} = \{x = (s, \lambda)^T \in \mathbb{R}^2 : s, \lambda \in [-1.5, 1.5]\}$ is chosen such that it includes the initial and final points. After computing the Lyapunov exponents at several points in \mathcal{X} it is concluded that an averaging time of 4 is sufficient and that T_0 can be chosen as 0.1. The spectral gap $\Delta\mu$ is 2. So, condition (18) is satisfied with $\varepsilon_{\text{gap}} \approx 4^{-1}$. We verify that the finite-time subspaces converge sufficiently by computing them at several points within the region \mathcal{X} . The first part of Figure (2) shows the approximate stable

manifolds computed using the dichotomic basis method with different number of re-initializations and the actual stable manifold (computed by a line search on the initial costate value and forward integrating for 20 units of time with $\|x_{eq} - x(20)\| \leq 10^{-9}$). The second part of Figure (2) shows the difference between the actual stable manifold and the approximate stable manifolds computed using the dichotomic basis method. The error is of the order of 10^{-3} .

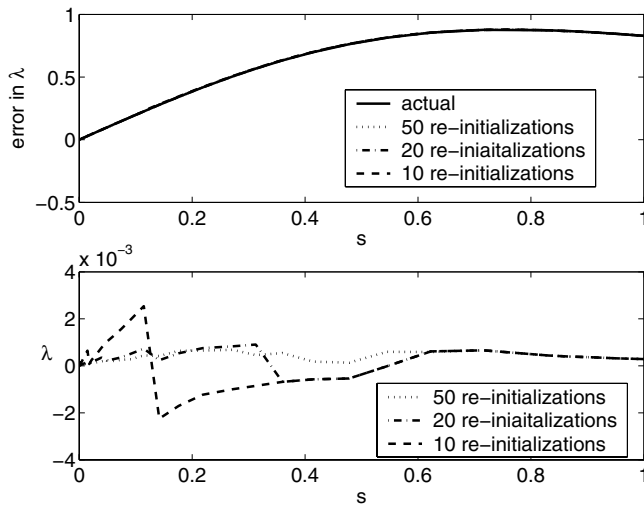


Fig. 2. The initial boundary-layer of example (2.1).

VII. CONCLUSIONS

The dichotomic basis method for solving completely hyper-sensitive optimal control problems has been modified by using Lyapunov exponents and vectors. It has been shown that the asymptotic Lyapunov vectors form dichotomic transformations that decouple the unstable dynamics from the stable dynamics. For numerical implementation, finite-time Lyapunov vectors are used to approximate the asymptotic Lyapunov vectors and to construct an approximate dichotomic basis. In this case, complete decoupling is not achieved. However, the dependence of the unstable modes on the stable ones can be reduced to a manageable level. A re-initialization process has been introduced to decrease the error accumulation. The new basis identifies the stable and unstable directions more accurately than the eigenvectors do though computation of the Lyapunov vectors requires more effort.

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