# Systems with parametric uncertainty

System with parametric uncertainty governed by

$$\dot{x}(t) = f(x(t), \delta)$$

The parameter $\delta$ is

- constant
- unknown
- known to take values on the bounded set $\Delta$

Assumption:

- For each $\delta \in \Delta$, the origin is an equilibrium point, i.e.,

$$f(0, \delta) = 0 \qquad \text{for all } \delta \in \Delta.$$

# ROA analysis for systems with parametric uncertainty

System with **constant parametric** uncertainty governed by

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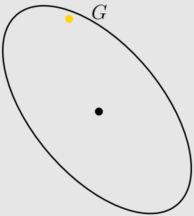
**Question:** Given a set $G$,

- is $G$ in the ROA for each $\delta \in \Delta$?
- is $G$ a subset of the robust ROA, defines as

$$\bigcap_{\delta \in \Delta} \{\zeta \in \mathbb{R}^n \; : \; \lim_{t \to \infty} \varphi(\zeta, t; \delta) = 0\}?$$

[ $\varphi(\zeta, t; \delta)$ is the solution at time $t$ with initial condition $\zeta$ for $\delta$.]



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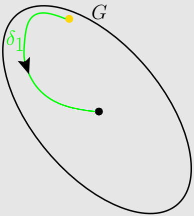

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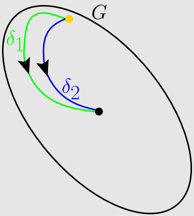

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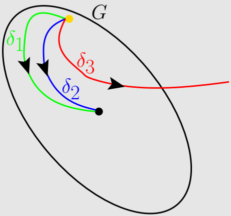

# ROA analysis for $\dot{x} = f(x, \delta)$

**Theorem:** If there exists a continuously differentiable function $V$ such that

- $V(0) = 0$, and $V(x) > 0$ for all $x \neq 0$
- $\Omega_{V,1} = \{x \ : \ V(x) \leq 1\}$ is bounded
- For each $\delta \in \Delta$, the set containment

$$\{x \ : \ V(x) \leq 1\} \backslash \{0\} \subset \{x \ : \ \nabla V(x) f(x, \delta) < 0\}$$

holds, then $\{x \in \mathbb{R}^n \ : \ V(x) \leq 1\}$ is an invariant subset of the robust ROA.

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**A few issues:**

- "For each $\delta \in \Delta$..." there are infinite number of set containment conditions.
- $V$ does not depend on $\delta$, though $f$ does, will this be restrictive?

# ROA analysis: $f(x, \delta)$ affine in $\delta$

Affine uncertainty dependence & bounded, polytopic $\Delta$ (with vertices $\mathcal{E}$)

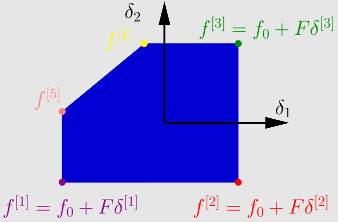
$$\dot{x}(t) = f_0(x(t)) + \sum_{i=1}^{m} f_i(x(t))\delta_i \quad = \quad f_0(x(t)) + F(x(t))\delta$$

**Theorem:** If $\Delta$ is a polytope, and for all $\delta \in \mathcal{E}$

$$\Omega_V \setminus \{0\} \subseteq \{x \in \mathbb{R}^n : \nabla V(x)(f_0(x) + F(x)\delta) < 0\},$$

then the set containment holds for all $\delta \in \Delta$.

**Proof:**
For each $\tilde{\delta} \in \Delta$, $\nabla V(x)F(x)\tilde{\delta}$ is a convex combination of $\{\nabla V(x)F(x)\delta \ : \ \delta \in \Delta\}$.

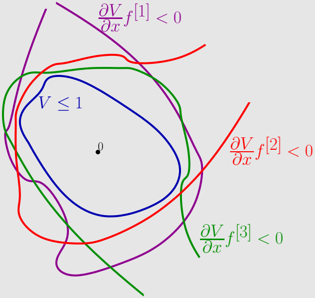


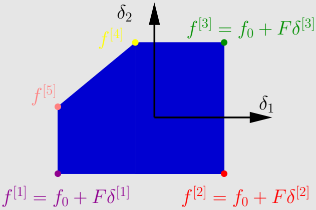
# ROA analysis with parameter-independent $V$ (2)

$$\dot{x}(t) = f_0(x(t)) + F(x(t))\delta$$

Impose at the vertices of $\Delta$, then they hold everywhere on $\Delta$.

$$\Omega_V \setminus \{0\} \subseteq \{x \in \mathbb{R}^n : \nabla V(x)(f_0(x) + F(x)\delta) < 0\}$$




For every $i = 1, \ldots, N_{vertex}$ (index to elements of $\mathcal{E}$),

$$-\left[(1-V)s_2 + s_3 \nabla V \cdot (f_0 + F\delta^{[i]}) + l_2\right] \text{ is SOS in } x \text{ (only)}$$

# SOS problem for robust ROA computation

$$\max_{0<\gamma, 0<\beta, V\in\mathcal{V}, s_1\in\mathcal{S}_1, s_{2\delta}\in\mathcal{S}_2, s_{3\delta}\in\mathcal{S}_3} \beta \quad \text{subject to}$$

$$s_{2\delta} \in \Sigma[x], \text{ and } s_{3\delta} \in \Sigma[x]$$

$$-[(\gamma - V)s_{2\delta} + \nabla V(f_0 + F(x)\delta)s_{3\delta} + l_2] \in \Sigma[x] \quad \forall \delta \in \mathcal{E},$$

$$-[(\beta - p)s_1 + V - 1] \in \Sigma[x]$$

- Bilinear optimization problem
- SOS conditions:
  - only in $x$
  - $\delta$ does not appear, but...
  - there are a lot of SOS constraints ($\delta \in \mathcal{E}$)

# Example

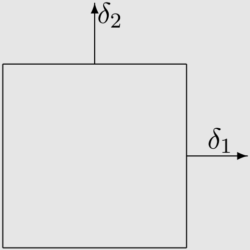
Consider the system with a single uncertain parameter $\delta$

$$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= -x_2 - (\delta + 2)(x_1 - x_1^3)\end{aligned}$$

with $\delta \in [-1, 1]$.

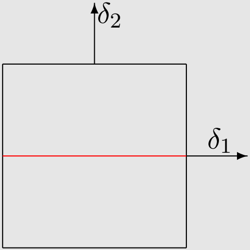
Codepad Demo: RobustROACalc.m and RobustROACalc.html

# Dealing with conservatism: partition $\Delta$




For all $\delta \in \Delta$:

$$\{x : V_0(x) \le 1\} \backslash \{0\} \subset \left\{ x : \frac{\partial V_0}{\partial x} f(x,\delta) < 0 \right\}$$




For all $\delta \in$ upper half of $\Delta$:

$$\{x : V_1(x) \le 1\} \backslash \{0\} \subset \left\{ x : \frac{\partial V_1}{\partial x} f(x,\delta) < 0 \right\}$$

For all $\delta \in$ lower half of $\Delta$:

$$\{x : V_2(x) \le 1\} \backslash \{0\} \subset \left\{ x : \frac{\partial V_2}{\partial x} f(x,\delta) < 0 \right\}$$

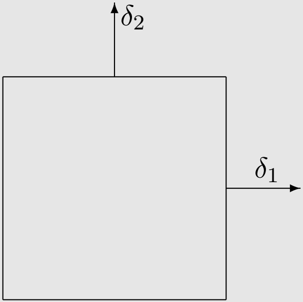
$V_1 := V_0$ and $V_2 := V_0$ are feasible for the right-hand side.
Improve the results by searching for different $V_1$ and $V_2$.

# Dealing with conservatism: branch-and-bound in $\Delta$

Systematically refine the partition of $\Delta$:

- Run an informal branch-and-bound (B&B) refinement procedure

Sub-division strategy: Divide the worst cell into $2$ subcells.



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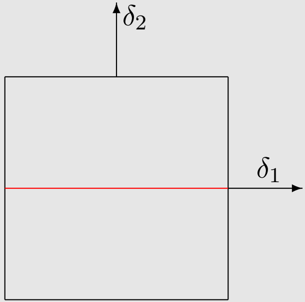

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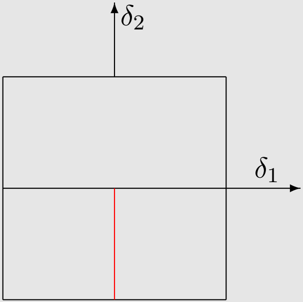

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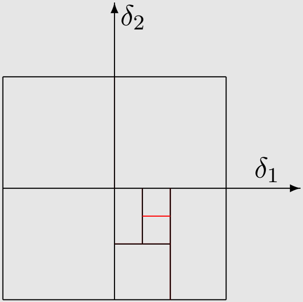
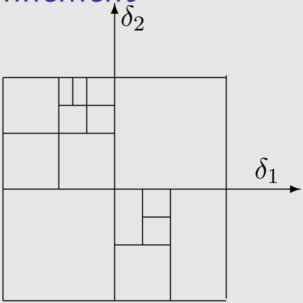


# Properties of the branch-and-bound refinement

- Yields piecewise-polynomial, $\delta$-dependent $V$.



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- Local problems are decoupled
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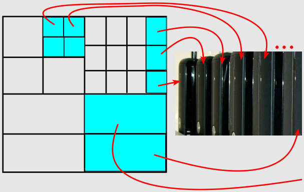


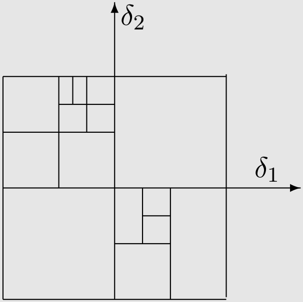

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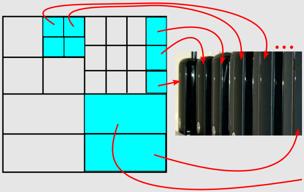
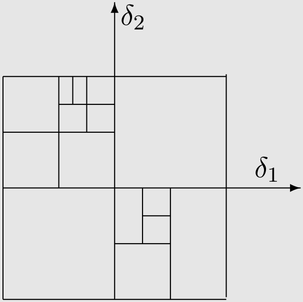



- Organizes extra info regarding system behavior: returns a data structure with useful info about the system
  - Lyapunov functions, SOS certificates,
  - certified $\beta$,
  - worst case parameters,
  - initial conditions for divergent trajectories,
  - values of $\beta$ not achievable, etc.

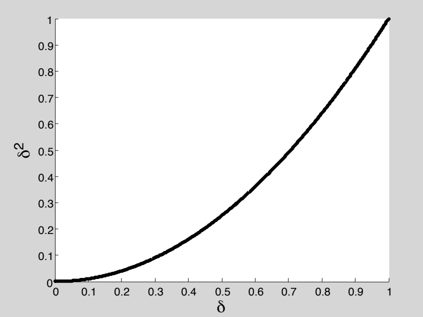
# Non-affine dependence on $\delta$

Let $g : \mathbb{R} \to \mathbb{R}$.

$$\begin{aligned} \dot{x}(t) &= f_0(x(t)) + \delta f_1(x(t)) + g(\delta) f_2(x(t)) \\ &= f_0(x(t)) + \delta f_1(x(t)) + \zeta f_2(x(t)) \end{aligned}$$

Treat $(\delta, g(\delta))$ as 2 parameters, whose values lie on a 1-dimensional curve. Then

∗ Cover 1-d curve with 2-polytope
∗ Compute ROA
∗ Refine polytope into a union of smaller polytopes
∗ Solve robust ROA on each polytope
∗ Intersect ROAs → robust ROA



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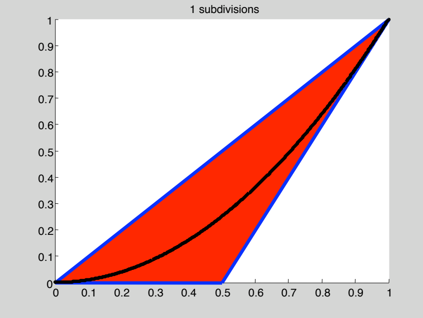

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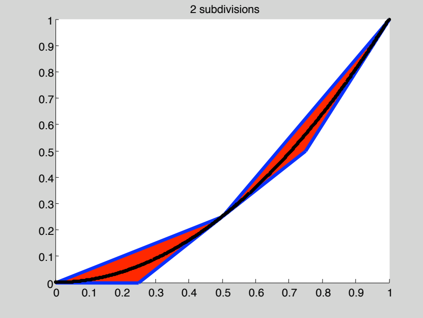

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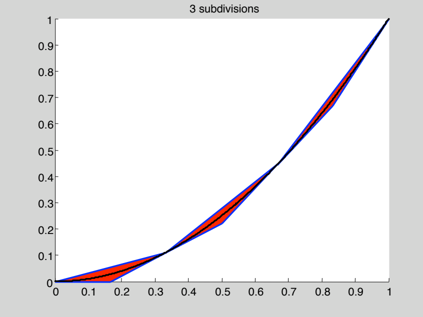


# Generalization of covering manifold

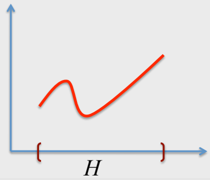
Given:

- polynomial $g(\delta)$ in many real variables, $\delta \in \mathbb{R}^q$
- domain $H \subseteq \mathbb{R}^q$, typically a polytope

Find a polytope that covers $\{(\delta, g(\delta)) : \delta \in H\} \subseteq \mathbb{R}^{q+1}$.

- Tradeoff between number of vertices, and
- excess "volume" in polytope

One approach: Find "tightest" affine upper and lower bounds to $g$ over $H$



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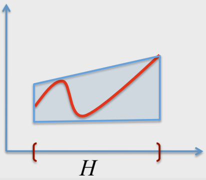

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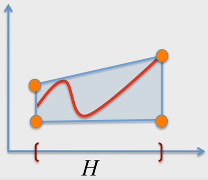

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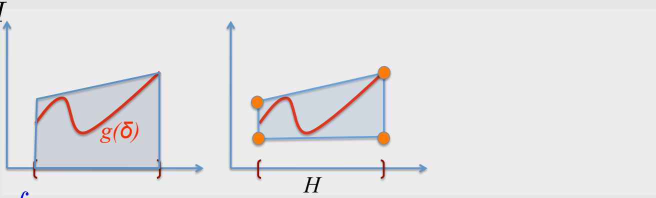


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An upper bound for this optimization can be computed through SOS optimization.

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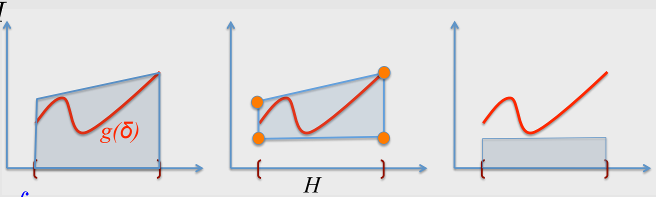


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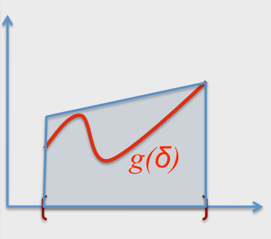
# Generalization of covering manifold (2)




$$\min_{c_0,c} \int_H (c_0 + c^T\delta)d\delta \quad \text{subject to}$$
$$c_0 + c^T\delta \geq g(\delta) \quad \forall \delta \in H$$

- $\int_H (c_0 + c^T\delta)d\delta$ is linear in $c_0$ and $c$.
- Let $H = \{\zeta \in \mathbb{R}^m \ : \ h_i(\zeta) \geq 0, \ i = 1, \ldots, N\}$ be an inequality description of $H$ where $h_1, \ldots, h_N$ are polynomials.

An upper bound on the optimal value of the problem above can be computed through the SOS program

$$\min_{c_0,c,\sigma_i \in \mathcal{S}_i} \int_H (c_0 + c^T\delta)d\delta \quad \text{subject to} \quad \sigma_1, \ldots, \sigma_N \in \Sigma[\delta]$$
$$-g(\delta) + (c^T\delta + c_0) - \sum_{i=1}^N \sigma_i(\delta)h_i(\delta) \in \Sigma[\delta].$$

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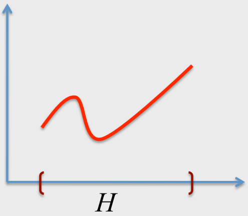

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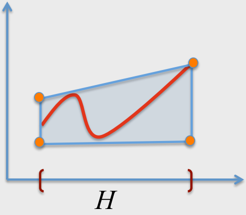

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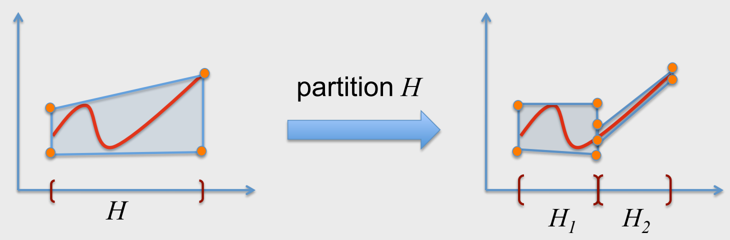


B&B refinement reduces the conservatism due to covering by reducing the extra covered space.

# Multiple non-affine parametric uncertainty

For multivariable $g$,

$$\dot{x} = f_0(x) + \delta_1 f_1(x) + \cdots + \delta_q f_q(x) + \\ g_1(\delta) f_{q+1}(x) + \cdots + g_m(\delta) f_{q+m}(x)$$

On $H$, bound each $g_i$ with affine functions $c_i$ and $d_i$

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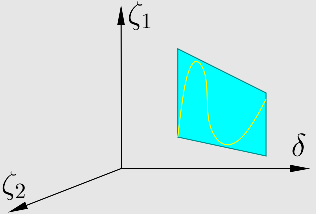

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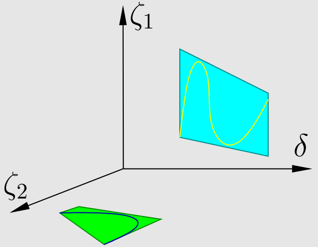

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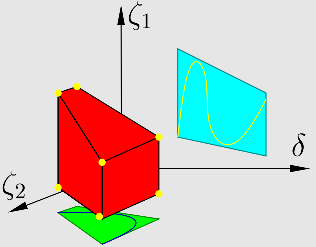
Then (Amato, Garofalo, Gliemo) a polytope covering $\{(\delta, g(\delta)) : \delta \in H\}$ is

$$\left\{(\delta, v) \in \mathbb{R}^{q \times m} : \delta \in H, C(\delta) \leq v \leq D(\delta)\right\}$$

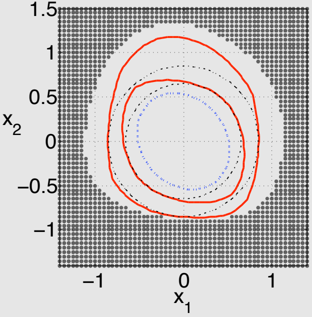
with $2^{q+m}$ easily computed vertices.



# Example: 2-state uncertain dynamics [Chesi, 2004]

$$\dot{x} = \begin{bmatrix} -x_1 \\ 3x_1 - 2x_2 \end{bmatrix} - \begin{bmatrix} 6x_2 - x_2^2 - x_1^3 \\ 10x_1 - 6x_2 - x_1 x_2 \end{bmatrix} \delta + \begin{bmatrix} 4x_2 - x_2^2 \\ 12x_1 - 4x_2 \end{bmatrix} \delta^2,$$

- $\delta \in [0, 1]$.
- No common quadratic $V$ for uncertain linearized dyn.
- $p(x) = x^T x$.
- $50$ branch-and-bound refinements




Blue dotted curve: Result from Chesi, 2004.

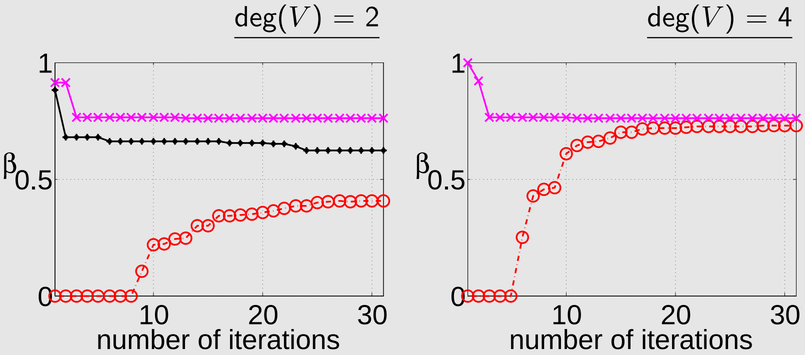
Red curves: Intersection of $\Omega_{V,1}$ for $V$'s obtained through the B&B refinement (inner for deg($V$) = 2 and outer for deg($V$) = 4)

Black dotted curves: Certified $\Omega_{p,\beta}$ for deg($V$) = 2 (inner) and for deg($V$) = 4 (outer)

Gray dots: Initial conditions of divergent trajectories for some $\delta \in [0, 1]$

# Example: 2-state uncertain dynamics

**B&B iterations:** Divide the cell with the smallest $\beta$ into 2.



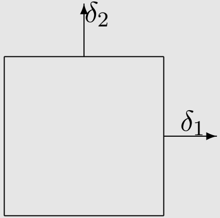

- Upper bounds from divergent trajectories
  - Upper bound does not depend on the complexity/degree of V
- Upper bounds from infeasibility of the affine relaxation
  - These show how the basis choice for $V$ impacts what is certifiable.
- Certified values (using ideas from previous 100+ slides)

# Dealing with large number of constraints

The SOS problem for the robust ROA includes the constraint:

$$-[(\gamma - V)s_{2\delta} + \nabla V(f_0 + F(x)\delta)s_{3\delta} + l_2] \in \Sigma[x] \quad \forall \delta \in \mathcal{E}$$

The number of vertices grows fast with the dimension of the uncertainty space.



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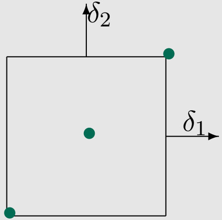


Suboptimal procedure:

- Sample $\Delta$ with fewer points (fewer than in $\mathcal{E}$)
- Optimize $V$ for this restricted sampling
- Certify a value of $\beta$, using this $V$, at all vertices of $\Delta$

The last step involves solving decoupled smaller problems.

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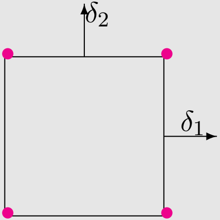


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- For each $\delta \in \mathcal{E}$, compute

$$\gamma_\delta := \max_{0<\gamma, s_{2\delta}\in\mathcal{S}_2, s_{3\delta}\in\mathcal{S}_3} \gamma \quad \text{subject to}$$
$$s_{2\delta} \in \Sigma[x], \text{ and } s_{3\delta} \in \Sigma[x]$$
$$-[(\gamma - \tilde{V})s_{2\delta} + \nabla\tilde{V}(f_0 + F\delta)s_{3\delta} + l_2] \in \Sigma[x],$$

and define

$$\gamma^{subopt} := \min\{\gamma_\delta \ : \ \delta \in \mathcal{E}\}.$$

$\Omega_{\tilde{V},\gamma^{subopt}}$ is an invariant subset of the robust ROA.

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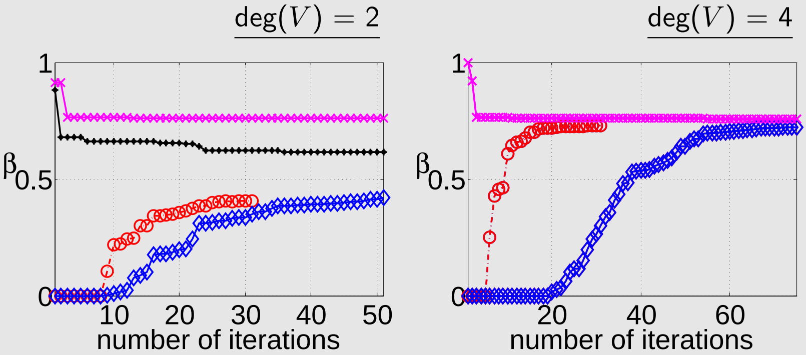
$$\gamma^{subopt} := \min\{\gamma_\delta \ : \ \delta \in \mathcal{E}\}.$$

  $\Omega_{\tilde{V},\gamma^{subopt}}$ is an invariant subset of the robust ROA.
- Determine the largest sublevel set of $p$ contained in $\Omega_{\tilde{V},\gamma^{subopt}}$

$$\max_{s_1\in\mathcal{S}_1,\beta} \beta \quad \text{subject to}$$
$$s_1 \in \Sigma[x]$$
$$-[(\beta - p)s_1 + \tilde{V} - \gamma^{subopt}] \in \Sigma[x].$$

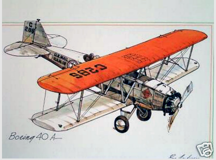
# Revisit *Chesi, 2004* with suboptimal $\Delta$ sampling

**B&B iterations:** Divide the cell with the smallest $\beta$ into 2.




- Upper bounds from divergent trajectories
- Upper bounds from infeasibility of the affine relaxation
- Lower bounds directly computing the robust ROA
- Lower bounds computing the robust ROA in two steps (sample $\Delta$ at cell center $\rightarrow$ optimize $V$ $\rightarrow$ verify at the vertices)

# Controlled aircraft [Short period pitch axis model]




Uncertain closed loop dynamics with

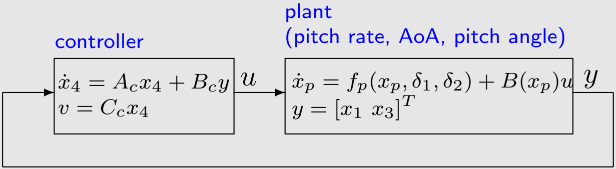
- $x = (x_p,\ x_4),\quad p(x) = x^T x$
- Cubic poly approx from Honeywell

$$\dot{x} = f_0(x) + f_1(x)\delta_1 + f_2(x)\delta_2 + f_3(x)\delta_1^2$$

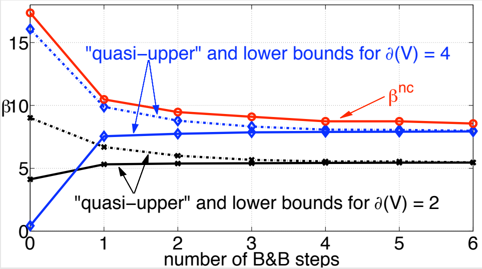
- $\delta_1 \in [0.99, 2.05]$ (uncertainty in the center of gravity)
- $\delta_2 \in [-0.1, 0.1]$ (uncertainty in mass)

Implemented on a 9-processor cluster

- Problems for $9$ cells are solved at a time
- Trivial speed up as expected.



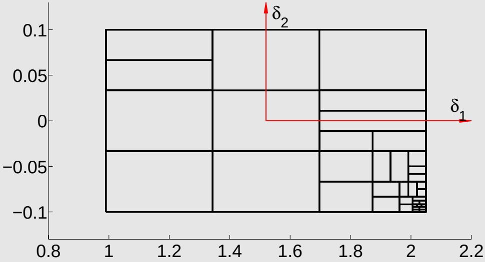
# Results - controlled aircraft dynamics




Strategy:

- Optimize at the center
- Verify at the vertices

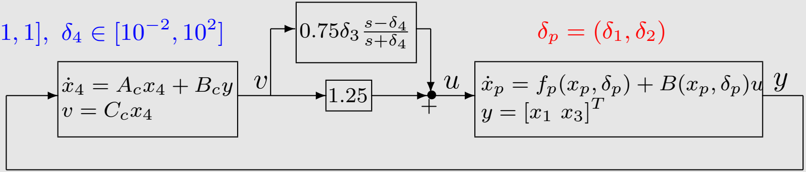
Quasi upper bound: $\beta$ certified (by the SOS problem) for the "center system" in the first step.




$$\dot{x} = f_0(x) + f_1(x)\delta_1 + f_2(x)\delta_2 + f_3(x)\delta_1^2$$

# Controlled aircraft + 1st order unmodeled dynamics

$\delta_3 \in [-1, 1]$, $\delta_4 \in [10^{-2}, 10^2]$ $\qquad$ $\delta_p = (\delta_1, \delta_2)$




$$\dot{x} = f_0(x) + \sum_{i=1}^{4} f_i(x)\delta_i + f_5(x)\delta_1^2 + f_6(x)\delta_1\delta_3 + f_7(x)\delta_2\delta_3$$

- First order LTI unmodeled dyn (state $x_5$)
- $p(x) = x^T x$, $x = \begin{bmatrix} x_p^T & x_4 & x_5 \end{bmatrix}^T$.

**Certified**

| dyn uncer \ param uncer | with | without |
|---|---|---|
| with | 2.8 | 4.9 |
| without | 5.4 | 8.0 |

How about other uncertainty descriptions (e.g. unmodeled dynamics)?

Coming up later

# Alternative uncertainty description

An alternate uncertainty description includes

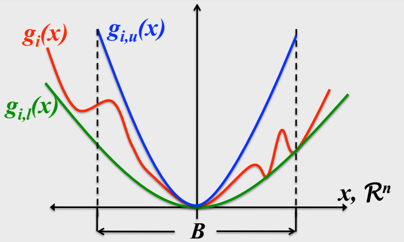
- nonpolynomial vector fields
- limited domain of validity

$$\dot{x}(t) = f_0(x(t)) + g(x(t))$$

$g : \mathbb{R}^n \to \mathbb{R}^n$ is unknown and satisfies polynomial, local bounds

$$g_l(x) \preceq g(x) \preceq g_u(x) \qquad \forall x \in B := \{x \ : \ b(x) \succeq 0\}$$

where $g_u,\ g_l \in \mathbb{R}[x]$, $g_u(0) = g_l(0) = 0$, and $B$ contains the origin.




Recall $v \preceq w$ implied $v_i \leq w_i$ for all $i = 1, \ldots, n$.

# Robust ROA with the alternative uncertainty description

A family $\mathcal{D}$ of vector fields:

$$\dot{x}(t) = f_0(x(t)) + g(x(t))$$

$g : \mathbb{R}^n \to \mathbb{R}^n$ is (only) known to satisfy

$$g_l(x) \preceq g(x) \preceq g_u(x) \quad \forall x \in B := \{x \ : \ b(x) \succeq 0\}$$

**Question:** Compute an estimate of the ROA for the uncertain vector field, i.e., a set that is

- invariant for each vector field of the form $f_0 + g \ \forall g \in \mathcal{D}$
- such that every trajectory with initial condition in the set converges to the origin.

Computed invariant subset of the robust ROA has to be a subset of $B$ (region of validity).

# Sufficient conditions in alternative uncertainty description

Impose the set containment constraint for each $g \in \mathcal{D}$

$$\{x \ : \ V(x) \leq 1\} \backslash \{0\} \subset \left\{ x \ : \ \frac{\partial V}{\partial x}(f_0(x) + g(x)) < 0 \right\},$$

then $\{x \in \mathbb{R}^n \ : \ V(x) \leq 1\}$ is an invariant subset of robust ROA.

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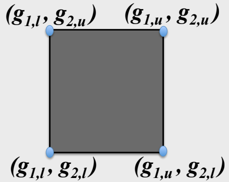
Vertices of "polytope of functions"
$\mathcal{E} := \{g \ : \ g_i = g_{i,\alpha} \quad \alpha = u, l\}$




Impose the set containment constraints for each $g \in \mathcal{E}$, then they will hold for each $g \in \mathcal{D}$

# Computing robust ROA (alternative uncer. model)

$$\max_{V \in \mathcal{V}, \beta > 0} \ \beta \ \text{ subject to}$$

$$V(0) = 0 \text{ and } V(x) > 0 \text{ for all } x \neq 0, \ \ \Omega_{V,1} \text{ is bounded,}$$

$$\Omega_{p,\beta} \subseteq \Omega_{V,1} \subseteq B,$$

$$\Omega_{V,1} \setminus \{0\} \subseteq \bigcap_{g \in \mathcal{E}} \{x \in \mathbb{R}^n : \nabla V(f_0(x) + g(x)) < 0\}.$$

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Let $B$ be defined by several polynomial inequalities $B = \{x \ : \ b(x) \succeq 0\}$. Then, a SOS relaxation for the above problem is

$$\max_{V\in\mathcal{V},\beta>0,s_1\in\mathcal{S}_1,\ s_{4k}\in\mathcal{S}_{4k},\ s_{2g}\in\mathcal{S}_{2g},s_{3g}\in\mathcal{S}_{3g}} \beta \ \text{ subject to}$$

$$V - l_1 \text{ is SOS},\ V(0) = 0,\ s_1,\ s_{41},\ldots,s_{4,m} \text{ are SOS}$$

$$s_{2g}, s_{3g} \text{ are SOS for } g \in \mathcal{E}$$

$$-[(\beta - p)s_1 + (V - 1)] \text{ is SOS}$$

$$b_k - (1 - V)s_{4k} \text{ is SOS for } k = 1,\ldots,m,$$

$$[(1 - V)s_{2\xi} + \nabla V(f_0 + g)s_{3\xi} + l_2] \text{ is SOS for } g \in \mathcal{E}$$

# Example

Consider the system governed by

$$\dot{x} = \begin{bmatrix} -2x_1 + x_2 + x_1^3 + 1.58x_2^3 \\ -x_1 - x_2 + 0.13x_2^3 + 0.66x_1^2 x_2 \end{bmatrix} + g(x),$$

where $g$ satisfies the bounds

$$-0.76x_2^2 \leq g_1(x) \leq 0.76x_2^2$$
$$-0.19(x_1^2 + x_2^2) \leq g_2(x) \leq 0.19(x_1^2 + x_2^2)$$

for all $x \in \{x \in \mathbb{R}^2 \ : \ x^T x \leq 2.1\}$.

- $p(x) = x^T x$
- deg($V$) = 4 (dashed curve)
- deg($V$) = 2 (solid curve)
- initial conditions for trajectories that leave the region of validity for $g(x) = \pm(0.76x_2^2, 0.19(x_1^2 + x_2^2))$ (dots)