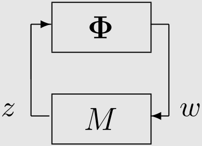
Recall: the small-gain theorem

For stable M and Φ, the feedback interconnection is internally stable if

$$\gamma(M)\gamma(\Phi) < 1.$$

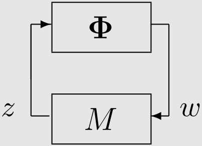


- γ is an upper bound on the global $\mathcal{L}_2 \to \mathcal{L}_2$ gain.
- Extensively used in linear robustness analysis where M is linear time-invariant (existence of global gains is guaranteed).

Recall: the small-gain theorem

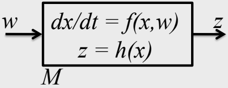
For stable M and Φ, the feedback interconnection is internally stable if

$$\gamma(M)\gamma(\Phi) < 1.$$



- γ is an upper bound on the global $\mathcal{L}_2 \to \mathcal{L}_2$ gain.
- Extensively used in linear robustness analysis where M is linear time-invariant (existence of global gains is guaranteed).
- How to generalize to nonlinear M with possibly only local gain relations?

Local small-gain theorems for stability analysis



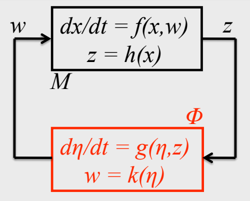
Let l be a positive definite function with $l(0) = 0$ e.g. $l(x) = \epsilon x^T x$ and $R > 0$.

For M: There exists a positive definite function V such that Ω_{V,R^2} is bounded and for all $x \in \Omega_{V,R^2}$ and $w \in \mathbb{R}^{n_w}$

$$\nabla V \cdot f(x,w) \leq w^T w - h(x)^T h(x) - l(x).$$

[M is "locally strictly dissipative" w.r.t. the supply rate $w^T w - z^T z$ certified by the storage function V.]

Local small-gain theorems for stability analysis



Let l be a positive definite function with $l(0) = 0$ e.g. $l(x) = \epsilon x^T x$ and $R > 0$.
Let $\tilde{l}$ be a positive definite function with $\tilde{l}(0) = 0$.

For M: There exists a positive definite function V such that Ω_{V,R^2} is bounded and for all $x \in \Omega_{V,R^2}$ and $w \in \mathbb{R}^{n_w}$

$$\nabla V \cdot f(x,w) \leq w^T w - h(x)^T h(x) - l(x).$$

[M is "locally strictly dissipative" w.r.t. the supply rate $w^T w - z^T z$ certified by the storage function V.]

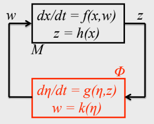
For Φ: There exists a positive definite function Q such that for all $\eta \in \mathbb{R}^{n_\eta}$ and $z \in \mathbb{R}^{n_z}$

$$\nabla Q \cdot g(\eta, z) \leq z^T z - k(\eta)^T k(\eta) - \tilde{l}(\eta).$$

[Φ is "strictly dissipative" w.r.t. $z^T z - w^T w$.]

Local small-gain theorems for stability analysis (2)

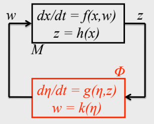
Conclusion: $S := V + Q$ is a Lyapunov function for the closed-loop for the closed-loop dynamics ($\dot{\xi} = F(\xi)$).



$$\xi = \begin{bmatrix} x \\ \eta \end{bmatrix}$$

Local small-gain theorems for stability analysis (2)

Conclusion: $S := V + Q$ is a Lyapunov function for the closed-loop for the closed-loop dynamics ($\dot{\xi} = F(\xi)$).

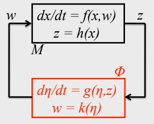


$$\xi = \begin{bmatrix} x \\ \eta \end{bmatrix}$$

Proof:

$$\begin{aligned} \nabla V \cdot f(x,w) &\leq w^T w - z^T z - l(x) && \forall x \in \Omega_{V,R^2} \ \& \ w \in \mathbb{R}^{n_w} \\ \nabla Q \cdot g(\eta,z) &\leq z^T z - w^T w - \tilde{l}(\eta) && \forall \eta \in \mathbb{R}^{n_\eta} \ \& \ z \in \mathbb{R}^{n_z} \end{aligned}$$

Local small-gain theorems for stability analysis (2)



Conclusion: $S := V + Q$ is a Lyapunov function for the closed-loop for the closed-loop dynamics ($\dot{\xi} = F(\xi)$).

$$\xi = \begin{bmatrix} x \\ \eta \end{bmatrix}$$

Proof:

$$\begin{aligned} \nabla V \cdot f(x,w) &\leq w^T w - z^T z - l(x) && \forall x \in \Omega_{V,R^2} \;\&\; w \in \mathbb{R}^{n_w} \\ \nabla Q \cdot g(\eta,z) &\leq z^T z - w^T w - \tilde{l}(\eta) && \forall \eta \in \mathbb{R}^{n_\eta} \;\&\; z \in \mathbb{R}^{n_z} \end{aligned}$$

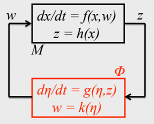
$$\nabla V \cdot f(x, g(\eta)) + \nabla Q \cdot g(\eta, h(x)) \leq l(x) + \tilde{l}(\eta)$$
$$\forall (x,\eta) \in \left\{(x,\eta) \;:\; V(x) + Q(\eta) \leq R^2\right\}$$

$$\nabla S \cdot F(\xi) \leq -l(x) - \tilde{l}(\eta) = -L(\xi)$$
$$\forall (x,\eta) \in \left\{(x,\eta) \;:\; S(x,\eta) \leq R^2\right\}$$

Corollary:

- $\{(x,\eta) \;:\; V(x) + Q(\eta) \leq R^2\}$ is an invariant subset of the ROA for the closed-loop dynamics.

Interpretation for the states x and (x, η)



$$\xi = \begin{bmatrix} x \\ \eta \end{bmatrix}$$

$\{(x, \eta) \ : \ V(x) + Q(\eta) \leq R^2\} = \{\xi \ : \ S(\xi) \leq R^2\}$ is an invariant subset of the ROA for the closed-loop dynamics ($\dot{\xi} = F(\xi)$).

Consequently,

- For $\eta(0) = 0$ and any $x(0) \in \Omega_{V,R^2}$, $x(t) \in \Omega_{V,R^2}$ for all $t \geq 0$ and $x(t) \to 0$ as $t \to \infty$.
- For any $\xi(0) = (x(0), \eta(0)) \in \Omega_{S,R^2}$, $x(t) \in \Omega_{V,R^2}$ for all $t \geq 0$ and $x(t) \to 0$ as $t \to \infty$.

Reiterating: For $\eta(0) = 0$, conclusions on x (in the first bullet above) hold even if Φ is not known but known to be strictly dissipative w.r.t. $z^T z - w^T w$.

Estimating the ROA (for x states)

Let p be a shape factor (as before) and $(\bar{V}, \bar{\beta}, \bar{R})$ be a solution to the above optimization

$$\max_{V \in \mathcal{V}, \beta \geq 0, R \geq 0} \quad \beta \quad \text{subject to}$$

$$V(x) > 0 \text{ for all } x \neq 0, \quad V(0) = 0,$$

$$\Omega_{p,\beta} \subseteq \Omega_{V,R^2},$$

$$\Omega_{V,R^2} \text{ is bounded},$$

$$\nabla V f(x,w) \leq w^T w - z^T z - l(x) \quad \forall\, x \in \Omega_{V,R^2}, \;\; \forall\, w \in \mathbb{R}^{n_w}.$$

If Φ is strictly dissipative w.r.t. $z^T z - w^T w$ and $\eta(0) = 0$, then for any $x(0) \in \Omega_{p,\bar{\beta}}$,

- $x(t)$ stays in $\Omega_{\bar{V},\bar{R}^2}$
- $x(t) \rightarrow 0$ as $t \rightarrow \infty$.

Estimating the ROA - SOS problem

Original problem:

$$\max_{V\in\mathcal{V},\beta\geq 0,R\geq 0} \quad \beta \quad \text{subject to}$$

$$V(x) > 0 \text{ for all } x \neq 0, \quad V(0)=0,$$

$$\Omega_{p,\beta} \subseteq \Omega_{V,R^2},$$

$$\Omega_{V,R^2} \text{ is bounded},$$

$$\nabla V f(x,w) \leq w^T w - z^T z - l(x) \quad \forall\, x \in \Omega_{V,R^2}, \;\; \forall\, w \in \mathbb{R}^{n_w}.$$

Use the S-procedure and standard relaxations to obtain a SOS reformulation:

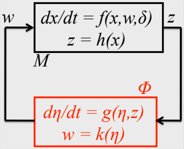
$$\max_{V\in\mathcal{V}_{poly},\beta\geq 0,R\geq 0,s_1\in\mathcal{S}_1,s_2\in\mathcal{S}_2} \beta \text{ subject to}$$

$$V - l_1 \in \Sigma[x], \quad s_1 \in \Sigma[x], \quad s_2 \in \Sigma[(x,w)],$$

$$(R^2 - V) - s_1(\beta - p) \in \Sigma[x],$$

$$-\nabla V f + w^T w - z^T z - l(x) - s_2(R^2 - V) \in \Sigma[(x,w)].$$

Incorporating parametric uncertainties in M



Uncertain parameters δ in the vector fields f can be handled as before.

- Restrict to polytopic Δ and affine δ dependence

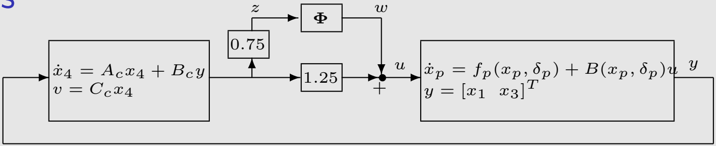
  $$\dot{x}(t) = f_0(x(t), w(t)) + F(x(t))\delta$$

  Resulting SOS condition is affine in δ

  $$-\nabla V(f_0(x,w)+F(x)\delta)+w^Tw-z^Tz-l(x)-s_2(R^2-V) \in \Sigma[(x,w)]$$

  and if it holds for the vertices of Δ then it holds for all $\delta \in \Delta$.
- Branch-and-bound in Δ
- Coverings for non-affine δ dependence and non-polytopic Δ

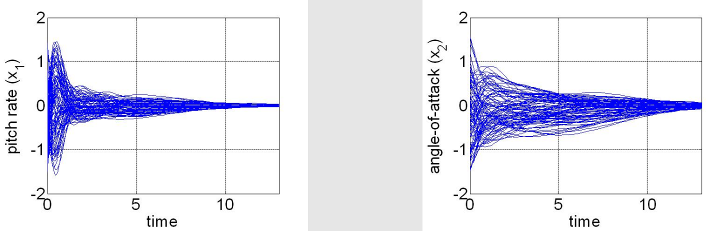
Example: Controlled aircraft dynamics with unmodeled dynamics



	no δ_p	with δ_p
no Δ	9.4 / 16.1	5.5 / 7.9
with Δ	4.2 / 6.7	2.4 / 4.1

In the table :
$(\partial(V) = 2/\partial(V) = 4)$

Closed-loop response with randomly generated first-order LTI Φ:



Relaxed version of the local small-gain theorem

Relax the strict dissipation for Φ by dissipation (i.e., $\tilde{l}(\eta) = 0$):

$$\nabla Q g(\eta, z) \leq z^T z - w^T w \qquad \forall \eta \in \mathbb{R}^{n_\eta} \;\; \& \;\; z \in \mathbb{R}^{n_z}$$

Weaker conclusion: For $\eta(0) = 0$ and for any $0 < \tilde{R} < R$,

- $x(0) \in \Omega_{V,\tilde{R}^2} \;\; \Rightarrow \;\; (x(t), \eta(t)) \in \Omega_{S,\tilde{R}^2} \quad \forall t \geq 0$
- $x(0) \in \Omega_{V,\tilde{R}^2} \;\; \Rightarrow \;\; x(t) \in \Omega_{V,\tilde{R}^2} \quad \forall t \geq 0 \;\; \& \lim_{t \to \infty} x(t) = 0$

Proof idea: arguments as before + Barbalat's lemma

Unit gain $\rightarrow$ Gains γ and $1/\gamma$

Conditions (that hold appropriately for x, η, w, z as indicated before)

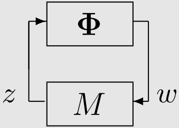
$$\begin{array}{ll} \nabla V f(x,w) \leq w^T w - z^T z - l(x) & \Rightarrow \|M\|_\infty < 1 \\ \nabla Q g(\eta,z) \leq z^T z - w^T w - \tilde{l}(\eta) & \Rightarrow \|\Phi\|_\infty < 1. \end{array}$$

These gain conditions can relaxed to gain of non-unity.

Previous results will hold when the dissipation inequalities are replaced by

$$\begin{array}{ll} \nabla V f(x,w) \leq w^T w - \gamma^2 z^T z - l(x) & (\Rightarrow \|M\|_\infty < 1/\gamma) \\ \nabla Q g(\eta,z) \leq \gamma^2 z^T z - w^T w - \tilde{l}(\eta) & (\Rightarrow \|\Phi\|_\infty < \gamma). \end{array}$$

Generalization to generic supply rates



Results hold when the "$\mathcal{L}_2$-gain supply rate" is replaced by a general supply rate.

Suppose that

- Φ is strictly dissipative w.r.t. the supply rate $r_1(z,w)$ with the corresponding storage function Q
- M satisfies

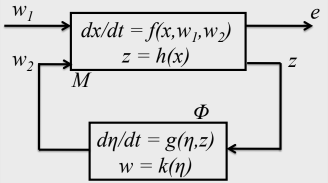
$$\nabla V f(x,w) \leq r_2(w,z) - l(x) \quad \forall x \in \Omega_{V,R^2} \ \& \ w \in \mathbb{R}^{n_w}$$

with

$$r_1(z,w) = -r_2(w,z) \quad \forall w,z.$$

Then, $\{(x,\eta) \ : \ V(x) + Q(\eta) \leq R^2\}$ is an invariant subset of the ROA for the closed-loop dynamics.

Local small-gain theorems for performance analysis

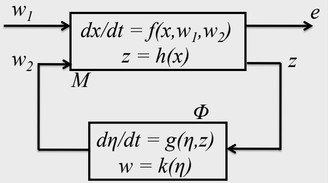


Global gain condition on Φ: starting from rest (i.e., initial condition equal to 0) Φ satisfies

$$\|w_2\|_2 = \|\Phi(z)\|_2 \leq \|z\|_2.$$

Goal: Find an upper bound on $\|e\|_2$ provided that M and Φ start from rest and $\|w_1\|_2 \leq R$.

Local small-gain theorems for performance analysis



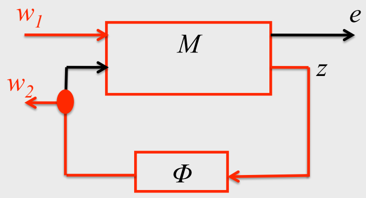
Global gain condition on Φ: starting from rest (i.e., initial condition equal to 0) Φ satisfies

$$\|w_2\|_2 = \|\Phi(z)\|_2 \leq \|z\|_2.$$

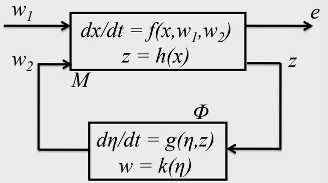
Goal: Find an upper bound on $\|e\|_2$ provided that M and Φ start from rest and $\|w_1\|_2 \leq R$.

Strategy:

- Bound $\|w_2\|_2$ in terms of $\|w_1\|_2$.



Local small-gain theorems for performance analysis



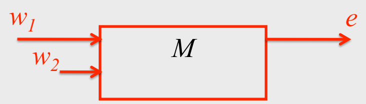
Global gain condition on Φ: starting from rest (i.e., initial condition equal to 0) Φ satisfies

$$\|w_2\|_2 = \|\Phi(z)\|_2 \leq \|z\|_2.$$

Goal: Find an upper bound on $\|e\|_2$ provided that M and Φ start from rest and $\|w_1\|_2 \leq R$.

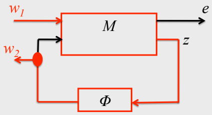
Strategy:

- Bound $\|w_2\|_2$ in terms of $\|w_1\|_2$.
- Bound $\|e\|_2$ in terms of $\|w_1\|_2$.



- Each step is a separate local gain analysis.

Step 1 (bound $\|w_2\|_{2,T}$ in terms of $\|w_1\|_{2,T}$):

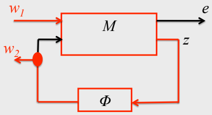


For $R > 0$, $0 < \alpha < 1$ and $\beta > 0$, if $\exists\ \mathcal{C}^1$ function V s. t. $V(0) = 0$, $V(x) > 0\ \forall x \neq 0$, Ω_{V,R^2} is bounded,

$$\nabla V f(x, w_1, w_2) \leq \beta^2 w_1^T w_1 + w_2^T w_2 - \frac{1}{\alpha^2} z^T z$$

$\forall x \in \Omega_{V,R^2}$, $w_1 \in \mathbb{R}^{n_{w_1}}$, and $w_2 \in \mathbb{R}^{n_{w_2}}$, then for Φ starting from rest and for all $T \geq 0$

$$x(0) = 0\ \&\ \|w_1\|_{2,T} \leq R/\beta \qquad \Rightarrow \qquad \|w_2\|_{2,T} \leq \alpha R/\sqrt{1-\alpha^2}.$$

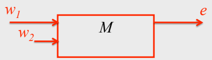


Step 1 (bound $\|w_2\|_{2,T}$ in terms of $\|w_1\|_{2,T}$):
For $R > 0$, $0 < \alpha < 1$ and $\beta > 0$, if $\exists\ \mathcal{C}^1$ function V s. t. $V(0) = 0$, $V(x) > 0\ \forall x \neq 0$, Ω_{V,R^2} is bounded,

$$\nabla V f(x, w_1, w_2) \leq \beta^2 w_1^T w_1 + w_2^T w_2 - \frac{1}{\alpha^2} z^T z$$

$\forall x \in \Omega_{V,R^2}$, $w_1 \in \mathbb{R}^{n_{w_1}}$, and $w_2 \in \mathbb{R}^{n_{w_2}}$, then for Φ starting from rest and for all $T \geq 0$

$$x(0) = 0\ \&\ \|w_1\|_{2,T} \leq R/\beta \qquad \Rightarrow \qquad \|w_2\|_{2,T} \leq \alpha R/\sqrt{1-\alpha^2}.$$



Step 2 (bound $\|e\|_{2,T}$ in terms of $\|w_1\|_{2,T}$): In addition to above conditions, if $\exists\ \mathcal{C}^1$ function Q s.t. $Q(0) = 0$, $Q(x) > 0\ \forall x \neq 0$, and

$$\nabla Q f(x, w_1, w_2) \leq \beta^2 w_1^T w_1 + w_2^T w_2 - \frac{1}{\gamma^2} e^T e$$

$\forall x \in \Omega_{Q,R^2/(1-\alpha^2)}$, $w_1 \in \mathbb{R}^{n_{w_1}}$, $w_2 \in \mathbb{R}^{n_{w_2}}$, then for $\mathbf{\Phi}$ starting from rest and for all $T \geq 0$

$$x(0) = 0\ \&\ \|w_1\|_{2,T} \leq R/\beta \qquad \Rightarrow \qquad \|e\|_{2,T} \leq \gamma R/\sqrt{1-\alpha^2}.$$