CDS 270 (Fall 09) - Assignment 6 (Due Tuesday, Nov. 10)

Exercise 1: The sum-of-squares based proof techniques we have covered are only applicable for systems with polynomial or rational vector fields. If the vector field of interest is neither polynomial not rational, then a pragmatic approach is to approximate or "cover" it by polynomial/rational vector fields over bounded regions of the state space. Another approach, which is applicable in certain cases, is *re-casting* the vector field into a polynomial or rational one. Consider the case

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= -\sin x. \end{aligned}$$

Now introduce new variables $z := \sin x$ and $\eta := \cos x$. Then, the above system can be re-written as

$$\begin{aligned} \dot{z} &= \eta y \\ \dot{y} &= -z \\ \dot{\eta} &= -zy \end{aligned}$$

subject to the constraint

$$z^2 + \eta^2 = 1.$$

- Discuss why these two system descriptions are equivalent.
- Recast the following vector field into an equivalent polynomial rational vector field subject to polynomial equality and/or inequality constraints.

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= \varphi(x_1 + x_2), \end{aligned}$$

  where

$$\varphi(\sigma) := \frac{\sigma}{\sqrt{1+\sigma^2}}.$$

Exercise 2: In the region of attraction analysis, for a given Lyapunov function candidate and a "small" positive constant ϵ, we tried to find the largest value of γ such that

$$\{x \,:\, V(x) \leq \gamma\} \subset \left\{x \,:\, \dot{V}(x) \leq -\epsilon x^T x\right\}$$

(assuming the system has an asymptotically stable equilibrium point at the origin). A related (but not equivalent) way of asking a similar question is

$$\gamma^* := \min_{x \neq 0} \quad V(x) \quad \text{subject to} \quad \dot{V}(x) = 0.$$

Let $\epsilon > 0$, V satisfy $V(x) \geq \epsilon x^T x$ for all $x \in \mathbb{R}^n$, and γ^* be defined as above. Then, for any $\gamma < \gamma^*$, the connected component of $\{x \,:\, V(x) \leq \gamma\}$ including the origin is an invariant subset of the region of attraction.

- Define

$$\mu^* := \max_{\mu>0} \quad \mu \quad \text{such that } V(x) \geq \mu \ \text{ for all } x \in \{x \ : \ \dot{V}(x) = 0\}.$$

Show that $\mu^* \leq \gamma^*$.

- Consider the problem

$$\mu_S := \max_{\mu>0,\ r\in\mathbb{R}[x]} \quad \mu \quad \text{such that} \ \ (V(x) - \mu) \cdot \left(x_1^{2d} + \cdots + x_n^{2d}\right) - \dot{V}(x) \cdot r(x) \in \Sigma[x], \tag{1}$$

where d is a positive integer such that

$$deg(V) + 2d \geq deg(\dot{V}) + \deg(r).$$

That is, once you pick the degree for the multiplier r, you will pick an integer d so that the last inequality holds. "*deg*" denotes the degree of its argument. Show that $\mu_S \leq \mu^*$.

Note that the optimization problem (1) can be solved without any bisection (line search) unlike the SOS optimization problems that we obtained for the maximization of γ subject to $\{x \ : \ V(x) \leq \gamma\} \subset \left\{x \ : \ \dot{V}(x) \leq -\epsilon x^T x\right\}$.

- Compute a Lyapunov functions candidate for the linearized dynamics of the Van der Pol system

$$\dot{x}_1 = -x_2$$
$$\dot{x}_2 = x_1 + (x_1^2 - 1)x_2.$$

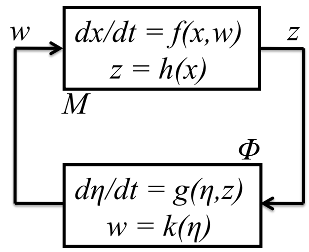
Implement the SOS problem in (1) to compute lower bounds for μ^* for this Lyapunov function candidate. Try different degrees for the multiplier r (and figure out the corresponding values for d).

- Use `pcontain` to implement the usual relaxation for $\{x \ : \ V(x) \leq \gamma\} \subset \left\{x \ : \ \dot{V}(x) \leq -\epsilon x^T x\right\}$ for the same Layunov function candidate from the previous part. Compare your results (maybe using `pcontour`).

Take $\epsilon = 10^{-6}$. The problem in (1) has a strict inequality constraint on μ. Replace this by $\mu \geq \epsilon$ (remember numerical optimization tools do not really distinguish between strict and non-strict inequalities).

Exercise 3: This is an open-ended question. Please get help from either `utopcu@cds.caltech.edu` or `andyl@cds.caltech.edu` if you seem to get stuck.

Recap - local small-gain type theorem: Consider the interconnection of two input-output systems M and Φ. Let x, η, w, and z be in $\mathbb{R}^{n_x}$, $\mathbb{R}^{n_\eta}$, $\mathbb{R}^{n_w}$, and $\mathbb{R}^{n_z}$, respectively.



Let $l_1 : \mathbb{R}^{n_x} \to \mathbb{R}$ be a positive definite function with $l_1(0) = 0$ e.g. $l_1(x) = \epsilon_1 x^T x$ and $R > 0$.
Let $\mathbb{R}^{n_\eta} \to \mathbb{R}$ be a positive definite function with $l_2(0) = 0$ e.g. $l_2(x) = \epsilon_2 x^T x$

Let $r : \mathbb{R}^{n_w} \times \mathbb{R}^{n_z} \to \mathbb{R}$.

Consider that the following conditions hold

For M: There exists a positive definite function V such that Ω_{V,R^2} is bounded and for all $x \in \Omega_{V,R^2}$ and $w \in \mathbb{R}^{n_w}$

$$\nabla V \cdot f(x,w) \leq r(w,z) - l_1(x).$$

For Φ: There exists a positive definite function Q such that for all $\eta \in \mathbb{R}^{n_\eta}$ and $z \in \mathbb{R}^{n_z}$

$$\nabla Q \cdot g(\eta, z) \leq -r(w,z) - l_2(\eta).$$

Then, $\{(x,\eta) \ : \ V(x) + Q(\eta) \leq R^2\}$ is an invariant subset of the region of attraction for the closed-loop dynamics. In particular, for $\eta(0) = 0$ and any $x(0) \in \Omega_{V,R^2}$, $x(t) \in \Omega_{V,R^2}$ for all $t \geq 0$ and $x(t) \to 0$ as $t \to \infty$.

We will assume that a suitable Q for Φ has already been constructed and we will try to "enlarge" the estimate of the region of attraction by enlarging the set Ω_{V,R^2} subject to the above constraints on V. For $r(w,z) = w^T w - z^T z$, this can be imposed through the following optimization (where $p : \mathbb{R}^{n_x} \to \mathbb{R}$ is a shape function just as before in the region of attraction analysis - for this exercise just take it as $p(x) = x^T x$).

$$\max_{V \in \mathcal{V}, \beta \geq 0, R \geq 0} \quad \beta \quad \text{subject to}$$
$$V(x) \geq l_1(x) \text{ for all } x \in \mathbb{R}^{n_x}, \quad V(0) = 0,$$
$$\Omega_{p,\beta} \subseteq \Omega_{V,R^2},$$
$$\Omega_{V,R^2} \text{ is bounded},$$
$$\nabla V \cdot f(x,w) \leq w^T w - z^T z - l_1(x) \quad \forall\ x \in \Omega_{V,R^2}, \quad \forall\ w \in \mathbb{R}^{n_w}.$$

If $f(x,w)$ and $h(x)$ are of the form $f(x,w) = \frac{F(x,w)}{G(x)}$ and $g(x) = \frac{H(x)}{G(x}$ with $G(x) > 0$ for all $x \in \mathbb{R}^{n_x}$, where F and H are vector of polynomials of their arguments and G is polynomial of its arguments, the above problem can be equivalently written as

$$\max_{V\in\mathcal{V},\beta\geq 0,R\geq 0} \quad \beta \quad \text{subject to}$$
$$V(x)\geq l_1(x) \text{ for all } x\in\mathbb{R}^{n_x}, \quad V(0)=0,$$
$$\Omega_{p,\beta}\subseteq\Omega_{V,R^2},$$
$$\Omega_{V,R^2} \text{ is bounded,}$$
$$\nabla V(x)\cdot F(x,w)\cdot G(x)\leq G(x)^2\cdot(w^Tw-z^Tz-l_1(x)) \quad \forall\, x\in\Omega_{V,R^2}, \;\; \forall\, w\in\mathbb{R}^{n_w}. \tag{2}$$

Note that $\nabla V(x)\cdot F(x,w)\cdot G(x)\leq G(x)^2\cdot(w^Tw-z^Tz-l_1(x))$ is a polynomial in x and w. The S-procedure and SOS relaxation for the problem in (2) is bilinear and therefore we will implement the following coordinate-wise affine iterations to compute sub-optimal solutions for this problem.

A useful side note: `pcontain` seems to handle the following question correctly. Given functions $f:\mathbb{R}^{n_1}\to\mathbb{R}$ and $g:\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}\to\mathbb{R}$,

$$\max_{\gamma} \quad \gamma \quad \{x\;:\;f(x)\leq\gamma\}\subseteq\{x\;:\;g(x,w)\leq 0\} \quad \text{for all } w\in\mathbb{R}^{n_2}.$$

Specifically, it seems to solve the following problem

$$\max_{\gamma,\; s\in\mathcal{S}} \quad \gamma \quad s\in\Sigma[(x,w)] \;\text{ and }\; -g(x,w)-s(x,w)\cdot(\gamma-f(x))\in\Sigma[(x,w)],$$

where $\mathcal{S}$ is a prescribed subset of polynomials (e.g. all polynomials of degree 2). □

<u>Initialization</u>: Solve the following optimization problem

$$\max_{V\in\mathcal{V}_{poly},\alpha\in\mathbb{R},s_3\in\mathcal{S}_3} \alpha \text{ subject to}$$
$$-\nabla V(x)\cdot F(x,w)\cdot G(x)+\big(w^Tw-z^Tz-l_1(x)\big)\cdot G(x)^2-s_3(x,w)\cdot(\alpha-x^Tx)\in\Sigma[(x,w)],$$

where $\mathcal{V}_{poly}$ and $\mathcal{S}_3$ are prescribed subsets of polynomials (.e.g. the set of polynomials that you want to use in the parameterization of V and s_3).

Use V from the solution of this problem to initialize the following iterations. You may need to write your own little line-search (on α for this step). Can you explain why the solution of this problem will be feasible for the problem in (2)?

<u>Step 1</u>: For fixed V, solve the following optimization problem

$$\max_{R\geq 0} \quad R^2 \text{ subject to}$$
$$\{x\;:\;V(x)\leq R^2\}\subseteq\big\{x\;:\;-\nabla V\cdot F\cdot G+\big(w^Tw-z^Tz-l_1\big)\cdot G^2\geq 0\big\} \quad \text{for all } w\in\mathbb{R}^{n_w}.$$

You can use `pcontain` for this step. Call the multiplier returned by `pcontain` s_1.

Step 2: For fixed V and R (from the previous step), solve the following optimization problem.

$$\max_{\beta \geq 0} \beta \text{ subject to}$$
$$\{x \ : \ p(x) \leq \beta\} \subseteq \{x \ : \ V(x) \leq R^2\}.$$

You can use `pcontain` for this step. Call the multiplier returned by `pcontain` s_2.

Step 3: For fixed s_1, s_2, β, and R from the previous steps, solve the following feasibility problem for V.

$$\begin{gathered} V \in \mathcal{V}_{poly}, \\ V - l_1 \in \Sigma[x], \quad s_2 \in \Sigma[x], \quad s_1 \in \Sigma[(x,w)], \\ (R^2 - V) - s_2(\beta - p) \in \Sigma[x], \\ -\nabla V \cdot F \cdot G + \left(w^T w - z^T z - l_1\right) \cdot G^2 - s_1(R^2 - V) \in \Sigma[(x,w)]. \end{gathered}$$

Step 4: Repeat steps 1-3 until a stopping criterion is satisfied.

- Take $\epsilon_1 = 10^{-6}$ and test your implementation on the following system M:

$$\begin{aligned} \dot{x} &= 2w - f(x) - x \\ z &= f(x), \end{aligned} \tag{3}$$

where

$$f(x) = \frac{(1+\gamma)(x+1)}{1+\gamma(x+1)^2} - 1$$

and $\gamma = 3/2$.