CDS 270 (Fall 09) - Assignment 4 (Due Friday, Oct. 23)

1. Prove the following two statements.
   - The sets of *positive semidefinite* and *sum-of-squares* polynomials are *convex cones.*
   - A quadratic polynomial is positive semidefinite if and only of it is sum-of-squares.

2. Let $p(x_1, x_2) = x_1^2 x_2^4 + x_1^4 x_2^2 + 1 - 3x_1^2 x_2^2$.
   - Is p sum-of-squares?
   - Is $p(x_1, x_2) \cdot (x_1^2 + x_2^2)$ sum-of-squares?
   - Can you intuitively interpret the difference between the results in the first two parts? (Hint: Remember to use `issos`.)
   - How can we use the results of the first two parts of this question to conclude that p is positive semidefinite (even though it is not sum-of-squares)?

3. Let $f_1(x) = x_1 + x_2 + 1$, $f_2(x) = 19 - 14x_1 + 3x_1^2 - 14x_2 + 6x_1x_2 + 3x_2^2$, $f_3(x) = 2x_1 - 3x_2$, $f_4(x) = 18 - 32x_1 + 12x_1^2 + 48x_2 - 36x_1x_2 + 27x_2^2$, and $f = (1 + f_1^2 f_2)(30 + f_3^2 f_4)$.
   - Compute a lower bound on the global minimal value of f.
   - Compute a lower bound on the minimal value of f over the set $\{x \in \mathbb{R}^2 \; : \; 1 - (1 - x_1)^2 - (1 - x_2)^2 \geq 0\}$. Use the S-procedure and sosopt to set up a SOS program to solve this problem. Try polynomial multipliers (wherever you need them) of different degrees.

4. Remember from the lecture on Oct 13th: If there exists a polynomial that satisfies

$$\begin{array}{c} V(x) - \epsilon x^T x \in \Sigma[x], \qquad V(0) = 0, \\ -\frac{\partial V}{\partial x} f(x) - \epsilon x^T x \in \Sigma[x], \end{array} \qquad (1)$$

   then the system $\dot{x} = f(x)$, with $f(0) = 0$, is globally asymptotically stable around the origin. Let's take $\epsilon = 10^{-6}$ and

$$f(x) = \begin{bmatrix} -x_2 - 1.5x_1^2 - 0.5x_1^3 \\ 3x_1 - x_2 \end{bmatrix}. \qquad (2)$$

   - Can you construct a quadratic Lyapunov function that satisfies the above conditions? (Hint: You can try to modify the Matlab script from the demo file on Oct 13th - it is available at course web site.)
   - If you cannot find a quadratic Lyapunov function, try a 4th degree one. If you cannot find a 4th degree Lyapunov function, then increase the degree of the candidate Lyapunov functions until you find one. (Hint: 4th degree should work.)

5. You will use the data in lecture4Data.mat for this exercise. You can download this .mat file from the course web site. This file contains variables V and f. If you care, V is a Lyapunov function (obtained through some analysis that we will cover later in this course) computed for a system governed by $\dot{x} = f(x)$. Compute a lower bound on the optimal value of the following optimization problem.

$$\begin{array}{rc} & \max_{\mu > 0} \quad \mu \\ \text{subject to} & \{x \ : \ V(x) \leq 0.01\} \subseteq \{x \ : \ \frac{\partial V}{\partial x} \cdot f(x) \leq -\mu V\}. \end{array} \tag{3}$$