CDS 270 (Fall 09) - Assignment 3 (Due Friday, Oct. 16)

1. Express the following problems as semidefinite programs (SDPs).

   (a) Linear Program (LP)

$$\begin{array}{ll} \text{minimize} & c^T x + d \\ \text{subject to} & Gx \leq h \\ & Ax = b. \end{array}$$

   (b) Quadratic Program (QP)

$$\begin{array}{ll} \text{minimize} & (1/2)x^T P x + q^T x + r \\ \text{subject to} & Gx \leq h \\ & Ax = b, \end{array}$$

   where $P \succeq 0$.

   (c) Quadratically constrained quadratic program (QCQP)

$$\begin{array}{ll} \text{minimize} & (1/2)x^T P_0 x + q_0^T x + r_0 \\ \text{subject to} & (1/2)x^T P_i x + q_i^T x + r_i \leq 0, \quad i = 1, \ldots, m \\ & Ax = b, \end{array}$$

   where $P_i \succeq 0$ for $i = 0, \ldots, m$.

   (d) Second order cone program (SOCP)

$$\begin{array}{ll} \text{minimize} & f^T x \\ \text{subject to} & \|A_i x + b_i\|_2 \leq c_i^T x + d_i, \quad i = 1, \ldots, m \\ & Fx = g. \end{array}$$

   Hint (Schur Complement). If $A \in \mathbb{S}^n_{++}$, $C \in \mathbb{S}^m$, and $B \in \mathbb{R}^{n \times m}$, then

$$\begin{pmatrix} A & B \\ B^T & C \end{pmatrix} \succeq 0 \iff C - B^T A^{-1} B \succeq 0.$$

2. Let $A \in \mathbb{R}^{m \times n}$ and let $b \in \mathbb{R}^m$.

   (a) Formulate the l_∞ approximation problem

$$\text{minimize } \|Ax - b\|_\infty = \max\{|a_1^T x - b_1|, \ldots, |a_m^T x - b_m|\}$$

   as a linear program.

   (b) Formulate the l_4 approximation problem

$$\text{minimize } \|Ax - b\|_4 = \left(\sum_{i=1}^m (a_i^T x - b_i)^4\right)^{1/4}$$

   as a QCQP.

(c) Convert the QCQP from part 2b to an SDP.

(d) Define A and b by

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{pmatrix}, \qquad b = \begin{pmatrix} 10 \\ 10 \\ 0 \end{pmatrix}$$

and solve the convex programs from parts 2a and 2c using Yalmip.

3. (S-Procedure) Let $F_0, F_1, \dots, F_m \in \mathbb{S}^n$ be symmetric matrices, and consider the following statements.

(i) For all $x \in \mathbb{R}^n$, if

(ii) $x^T F_i x \geq 0$ for all $i \in \{1, \dots, m\}$ then $x^T F_0 x \geq 0$. $\{x \in \mathbb{R}^n : x^T F_0 x < 0, x^T F_1 x \geq 0, \dots, x^T F_m x \geq 0\} = \emptyset$.

(iii) There exist numbers $\tau_i \geq 0$ such that $F_0 \succeq \sum_{i=1}^m \tau_i F_i$.

(a) Prove that (i) $\Leftrightarrow$ (ii) $\Leftarrow$ (iii).

(b) Let

$$F_0 = \begin{pmatrix} -2 & -1 \\ -1 & 0 \end{pmatrix} \quad \text{and} \quad F_1 = \begin{pmatrix} -4 & -3 \\ -3 & -2 \end{pmatrix}$$

and solve the LMI from (iii) using Yalmip.