CDS 270-4 Spring 2008: Estimation and Control Over Wireless Sensor Network

Ling Shi

Reminder: The 10 problems in this homework set worth 60/100 of your total term score. The homework is due anytime before the last lecture. If you need extra time after the last lecture, send an e-mail to shiling@cds.caltech.edu for an extension of maximum 3 days.

Notations: The following notations are used for Problem 1-10 unless stated explicitly.

- $A, Q, X \in \mathbb{R}^{n\times n}, C \in \mathbb{R}^{m\times n}, R \in \mathbb{R}^{m\times m}, X \geq 0, Q \geq 0, R > 0$.
- $\lambda_i(A) \triangleq i$-th eigenvalue of A; $\text{Tr}(X) \triangleq$ trace of X; $\text{Det}(X) \triangleq$ determinant of X.
- $h(X) \triangleq AXA' + Q$; $g(X) \triangleq AXA' + Q - AXC'[CXC' + R]^{-1}CXA'$
- $\hat{x}_k^- \triangleq \mathbb{E}[x_k|$ given all measurement up to time $k-1]$.
- $\hat{x}_k \triangleq \mathbb{E}[x_k|$ given all measurement up to time $k]$.
- $e_k^- \triangleq x_k - \hat{x}_k^-$; $e_k \triangleq x_k - \hat{x}_k$;
- $P_k^- \triangleq \mathbb{E}[e_k^- e_k^{-'}]$; $P_k \triangleq \mathbb{E}[e_k e_k']$; $P_\infty^- \triangleq \lim_{k\to\infty} P_k^-$; $P_\infty \triangleq \lim_{k\to\infty} P_k$.
- $f^i(X) \triangleq \underbrace{f \circ \cdots \circ f}_{i \text{ times}}(X)$

Problem 1 Let $X \geq Y > 0$. Prove the following.

(a) $X[i:j, i:j] > 0$, where $1 \leq i \leq j \leq n$. [1 point]

(b) $CXC' \geq 0$ and $CXC' \geq CYC'$. [1 point]

(c) $\text{Tr}(X) > 0$, $\text{Det}(X) > 0$. [1 point]

(d) X is diagonalizable. [1 point]

(e) $0 < X^{-1} \leq Y^{-1}$. [2 points]

Problem 2 Assume $\lambda_i(A) \geq 1\, \forall\, i$. Prove the following.

(a) $h(X) \geq X$ does not always hold. [3 points]

(b) $\text{Tr}\big(h(X)\big) \geq \text{Tr}(X)$ always holds. [3 points]

Problem 3 Let $\rho(A) = \max_i |\lambda_i(A)|$. Show that $\rho(A) < 1$ if and only if there exists a $B > 0$ such that $B - A'BA > 0$. [6 points]

Problem 4 Consider the following system.

$$\begin{aligned} x_k &= Ax_{k-1} + w_{k-1} \\ y_k &= Cx_k + v_k \end{aligned}$$

Assume w_k and v_k are white gaussian with zero means and covariances $Q \geq 0$ and $R > 0$. Also assume (A, C) is detectable and $(A, \sqrt{Q})$ is stabilizable. Denote P_∞ as the steady state error covariance computed from a Kalman filter. Prove the following: [6 points]

$$P_\infty \leq h(P_\infty) \leq h^2(P_\infty) \leq h^3(P_\infty) \leq \cdots$$

Problem 5 "More sensors provide better estimation accuracy than less sensors". In the context of Kalman filtering, prove this for vector systems. Give necessary assumptions if needed. [6 points]

Problem 6 Read the paper "*Change Sensor Topology When Needed: How to Efficiently Use System Resources in Control and Estimation Over Wireless Networks*" by Ling Shi, Karl H. Johansson and Richard M. Murray, which is posted on the class webpage.

(a) Find an error in Section V-C where the properties of the *Tree Reconfiguration Algorithm* is proved. [2 points] (*Hint*: examine the P_∞ term)

(b) Correct the error that you find in part (a). [4 points]

Problem 7 Consider the following system dynamics

$$\begin{aligned} x_k &= 0.9x_{k-1} + w_{k-1} \\ y_k^1 &= x_k + v_k^1 \\ y_k^2 &= x_k + v_k^2 \\ y_k^3 &= x_k + v_k^3 \end{aligned}$$

where w_k and $v_k^i, i = 1, 2, 3$ are white gaussian with zero means and covariances $Q = 1, R_1 = 1.5, R_2 = 1, R_3 = 0.5$. The sensors form a line topology as shown in Figure 1.

The transmission energy is given as follows

$$e_i(d) = e, e_i(2d) = 4e, e_i(3d) = 8e.$$

For example, T_0 has a total transmission energy cost $3e$, while if T_0 is changed to T_{star}, i.e, all three sensors communicate with fusion center directly, the total cost is $13e$. Assume the following performance specification is received at the fusion center.

$$\begin{aligned} P_\infty &\leq 0.75, 1 \leq k \leq 100 \\ P_\infty &\leq 0.25, 101 \leq k \leq 200 \\ P_\infty &\leq 1.0, 201 \leq k \leq 300 \\ P_\infty &\leq 0.75, 301 \leq k \leq 500 \end{aligned}$$

(a) Using the sensor communication model introduced in the class (i.e, h hops introduce $h-1$ delays) to calculate the minimum energy subtrees which provide the corresponding estimation accuracy at the fusion center according to the above performance specification. [2 points]

(b) Simulate the system using Matlab from $k = 1$ to 500 and change the sensor trees according to what you find in part (a) that meets the performance specification. Plot the resulting $x_k, \hat{x}_k$, the resulting e_k and the result P_k in three subplots. Label the energy used for different trees on the P_k plot. [4 points]

Problem 8 Two sensors are available for estimating the following process

$$x_k = Ax_{k-1} + w_{k-1}$$

The measurement equations of sensor 1 and 2 are given as follows.

$$\begin{aligned} y_k^1 &= C_1 x_k + v_k^1 \\ y_k^2 &= \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} x_k + \begin{bmatrix} v_k^{21} \\ v_k^{22} \end{bmatrix} \end{aligned}$$

where w_{k-1}, v_k^1, v_k^{21} and v_k^{22} are zero mean and white gaussian with covariances $Q \geq 0, R_1 > 0, R_1 > 0$ and $R_2 > 0$ respectively.

An estimator that uses Kalman filter needs to select a sensor at each time k. The cost of using sensor 1 at each time is J_1, and using sensor 2 is J_2. The sensor selection process is as follows. At each time k, a uniform random number between 0 and 1 is generated. If

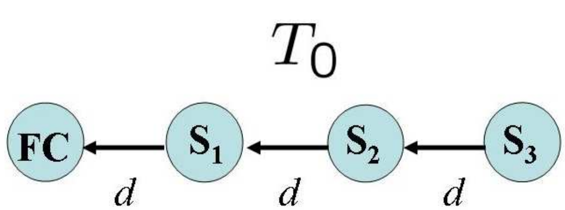

Figure 1: Line Topology

the number is less than α, sensor 1 is selected; otherwise, sensor 2 is selected. The long term average budget of the estimator is J_a, where $J_1 < J_a < J_2$. Solve the following sensor selection problem.

$$\min_{\alpha} \mathbb{E}[P_k]$$

$$\text{subject to} \qquad \alpha J_1 + (1-\alpha) J_2 \leq J_a$$

If closed form solution cannot be obtained, minimize $\mathbb{E}[P_k]$ as much as you can. [6 points] (*Hint*: find an upper bound of $\mathbb{E}[P_k]$ and minimize that upper bound instead)

Problem 9 Consider the following system

$$\begin{aligned} x_k &= Ax_{k-1} + w_{k-1} \\ y_k &= Cx_k + v_k \end{aligned}$$

where w_k and v_k are white gaussian with zero means and covariances $Q \geq 0$ and $R > 0$. The sensor can directly communicate to a fusion center with energy cost e_2 and delay 0; or it can communicate to the fusion center via a set of relay nodes, in which case, it only uses e_1 energy at each time, but with delay τ (see Figure 2). Assume the sensor has enough computational capability so that it runs a Kalman filter and sends $\hat{x}_k, P_k$ to the fusion center (or to the closest relay node).

Find the optimal sensor scheduling, i.e, when to use which route, so that the long term average of the energy cost is minimized, and $P_k \leq P_{\text{desired}}$ is guaranteed at the fusion center. Consider all possible cases and provide necessary assumptions to support your result. [6 points]

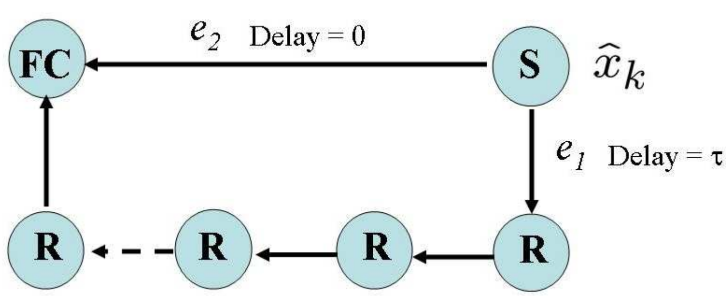

Figure 2: Sensor Scheduling

Problem 10 A vehicle is trying to track a given reference. The vehicle dynamics is given by

$$x_k = x_{k-1} + u_{k-1} + w_{k-1} \tag{1}$$

where w_k is the process noise assumed to be white gaussian with covariance

$$Q = \begin{bmatrix} 0.01 & 0 \\ 0 & 0.01 \end{bmatrix}.$$

Three range sensors are placed in the first quadrant to measure the vehicle state (see Figure 3). Denote S_p^i as the coordinate of sensor i which are chosen as follows:

$$S_p^1 = [\ 2 \quad 2\ ], \quad S_p^2 = [\ 4 \quad 4\ ], \quad S_p^3 = [\ 5 \quad 2\ ].$$

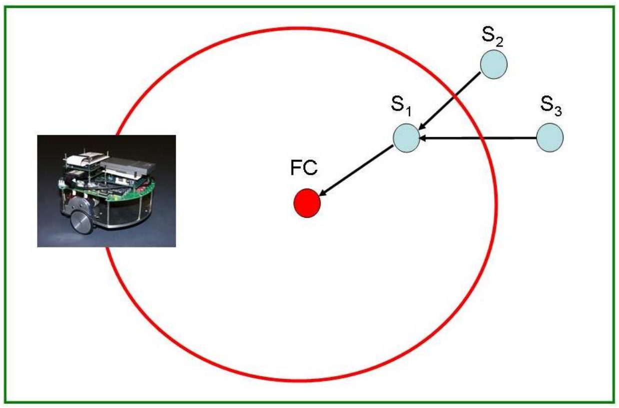

Figure 3: Reference Tracking

When S_i takes a measurement, it returns

$$y_k^i = ||x_k - S_p^i|| + v_k^i, \tag{2}$$

where v_k^i is white gaussian, zero mean and with covariances $\Pi_i = 0.2\,\forall\, i$. Assume the measurements from S_2 and S_3 are delayed one time due to the multi-hop communication.

A reference signal r_k is received at each time for the vehicle to track, which is generated as follows.

$$r_{k+1} = A_{\text{ref}} r_k, r_0 = [4 \ \ 0]$$

where

$$A_{\text{ref}} = \begin{bmatrix} \cos(0.02) & -\sin(0.02) \\ \sin(0.02) & \cos(0.02) \end{bmatrix}.$$

The following control law is used

$$u_k = r_k - \hat{x}_k$$

where $\hat{x}_k$ is generated via an Extended Kalman filter by linearing the measurement equation around $\hat{x}_k$.

(a) Implement the Extended Kalman filter and plot the running average of the estimation error and tracking error, from which you shall see that P_k does not converge. [3 points]

(b) Generate a video that shows:

- Sensor positions [which should remain fixed in your video]
- Reference received at each time k [which should dynamically change]
- Actual vehicle position at each time k [which should dynamically change]

When you turn in your homework, provide a snapshot of your video where $r_k = [-4\ \ 0]$ for the first time. [3 points]